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A BL-Algebra on Type-2 Fuzzy Sets

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ABSTRACT — In this paper, the formal properties of a peculiar BL-algebra whose elements are particular type-2 fuzzy sets are investigated. These type-2 fuzzy sets have membership degrees that are triangular fuzzy number totally ordered on [0, 1]. This algebra, endowed with suitable functions for data modelling, allows to handle a wide range of applications.

Keywords – Type-2 fuzzy sets, BL-algebras, Basic logic

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1. INTRODUCTION

BL-algebras have been introduced by Hajek [12] as those algebras which are appropriate for modelling formulas of Basic Logic. Specifically, they are the Lindenbaum algebras of the theories of Basic Logic. The latter aims at formalizing in a quite general way statements of fuzzy nature. BL-algebras may be understood as being residuated lattices fulfilling some further natural requirements. Moreover, BL-algebras form a variety of residuated lattices and they have been investigated in several papers, e.g. [1 ,4, 14, 19, 21]. In particular, by means of Basic Logic and BL-algebras, it is possible to build in a simple and elegant way a general theory of fuzzy logic in narrow sense.
Type-2 fuzzy sets, introduced by Zadeh [15] in 1975, have membership degrees that are themselves fuzzy sets. So a type-1 fuzzy set is a special case of a type-2 fuzzy set. Their properties and their applications have been studied in many papers, e.g. [2, 3, 13, 15, 16, 18].
In this paper a BL-algebra is introduced whose support set is the set of type-2 fuzzy sets having totally ordered triangular numbers on [0, 1] as membership degrees.
In [6, 10] the monoidal residuation properties are investigated, in this paper these results are extended by introducing lattice operations and BL-algebras properties.
The paper is organized as follows. In section 2 the mathematical concepts are given the following sections rely on, in section 3 the commutative monoid is introduced and a suitable order relation is studied in the next section. By means of this relation in section 5 the monoid is endowed with a residuation operation. In section 6 a BL-algebra (BL-chain) is defined. In section 7 the existence of a BL-algebra on the ordinal sum of BL-chains is proved. Finally, in section 8 the wide range of applications investigated so far are summarized and section 9 sketches some possible further investigation areas.

2. MAIN DEFINITIONS

A *commutative partially ordered monoid* [11] is a structure (L, *, e, ≤) such that (L, *, e) is a *commutative monoid*, where the element e is the unit, ≤ is a *partial order* on L and for all a, b, c, d ∈ L, if a ≤ b and c≤d then a*c ≤ b*d.

An algebra (L, ∩, ∪) is a *lattice* if the following identities are true in L:
Idempotency) x∩x = x, x∪x = x
Commutativity) x∩y = y∩x, x∪y = y∩x
Associativity) x∩(y∩z)=(x∩y)∩z, x∪(y∪z)=(x∪y)∪z
Absorption) x∩(x∪y) = x∪(x∩y) = x.

A *residuated lattice* [12] (L, ∩, ∩, *, ⇒, e, 0) is a structure such that:
i) (L, ∩, ∪, *, ⇒, e, 0) is a lattice with the greatest element e and the least element 0 (with respect to the lattice ordering ≤);
ii) (L, *, e) is a commutative monoid with the unit element e;

iii) $*$ and $\Rightarrow$ form an adjoint pair, i.e., for all a, b $\in$ L, $c*a \le b$ iff $c \le a \Rightarrow b$ (Galois relation). The binary operation $\Rightarrow$ on L is called *residuum*. The residuum $\Rightarrow$ is antitone in the left argument, monotone in the right element and for any a, b $\in$ L results $e \Rightarrow a = a$.

A residuated lattice (L, $\cap$, $\cup$, $*$, $\Rightarrow$, e, 0) is a *BL-algebra on L* [12] iff the following identities hold for any x,y $\in$ L:
i) $x \cap y = x * (x \Rightarrow y)$;
ii) $(x \Rightarrow y) \cup (y \Rightarrow x) = e$.

In each BL-algebra the following relation also holds:
$x \cup y = ((x \Rightarrow y) \Rightarrow y) \cap ((y \Rightarrow x) \Rightarrow x)$

Let A be a non empty classical set. A *fuzzy set* s on A is a function s: A --> [0, 1]. If a $\in$ A then s(a) is said the *membership degree* of a to A.

A *triangular fuzzy number* x=[a, b, c] on [0, 1] is a fuzzy set whose membership function is a triangle whose vertices are the points (a, 0), (b, 1) and (c, 0). In the sequel the following extended operations are used on the class of the [0,1]-triangular fuzzy numbers: i) $\alpha*[a,b,c]=[\alpha*a, \alpha*b, \alpha*c]$ (*product of a real number*); ii) [a, b, c] + [d, e, f] = [a+d, b+e, c+f] (*sum*).

A *type-2 fuzzy set* s_2 [20] on A is a function s_2: A --> $[0, 1]^{[0,1]}$.

3. The Commutative Monoid

Suppose that one has the following objects:

i) **U**: a finite universe of the discourse of cardinality p;

ii) **Tr** = {[0, 0, 0], [1, 1, 1]} $\cup$ {[a, b, c]: {a, b, c} $\subset$ [0, 1]}: a set of totally ordered triangular fuzzy numbers. [a, b, c] $\le$ [d, e, f] iff $a \le d$, $b \le e$, $c \le f$. It is worth noting that the crisp numbers: [0, 0, 0] and [1, 1, 1] belong to **Tr**;

iii) $\mathbf{F_2}$ = {a: $\Sigma_{i:\, m\ldots1,\ \text{with}\ m\le p}\, x_i/u_i$}: class of the type-2 fuzzy sets **U** --> **Tr**, where $x_i \in$ **Tr**, $x_i < x_{i+1}$, and $\{u_m, u_{m-1}, \ldots, u_1\}$ belongs to the class of crisp partitions ***P***(**U**) on **U**. In the sequel the elements u_i are called *crisp parts* and the elements x_i *fuzzy parts*

iv) **S(U)**= {[**0**= [0, 0, 0]/U, (1, 0, 1)], [**1**=[1,1,1]/U, (0, 1, 1)]} $\cup$ {[a, t]: a $\in$ $\mathbf{F_2}$, and $t=(k, s, a_m, a_{m-1}, \ldots, a_1)$ is a suitable t-uple of positive integers, that satisfies the following constraints: *j)* if k = 1 then a_i=1 for any i:1...m; *jj)* if k>1 the t-uple $(a_m, a_{m-1}, \ldots, a_1)$ is symmetric with respect to the central values }; *jjj)* s = 0 for **0**, instead s=1 for any A $\neq$ **0** and **1** in **S(U)**. Moreover $(k, s, a_m, \ldots, a_1) = (1, s, 1, 1, \ldots, 1)$ iff the related type-2 fuzzy set is not the product of other sets through the operation $\Diamond$ introduced in the sequel.

One can give the following intuitive meaning: the type-2 fuzzy set $\Sigma_{i:\, m\ldots1,\ \text{with}\ m\le p}\, x_i/u_i$ represents an *attribute* A in the sense that the elements $u_i \subseteq$ **U** *satisfy* A with strength x_i. Moreover, one says that the elements of U are *classified* with respect to A by means of the linguistic terms represented by the type-1 fuzzy sets $x_i \in [0,1]^{[0,1]}$. With this interpretation the element **0** and **1** are read as "*No information*" and "*Not compatible*", respectively. The label standing for "*No information*" is utilized when there is no information available about the elements in U in order to assess the degree they satisfy the attribute A with, whereas "*Not compatible*" is used if the elements in U are not compatible with the property A.

Given
$\mathbf{A=[\Sigma_{i:\, n\le p\ldots1}\, x_i/u_i,\ (k_A, s_A, a_n, a_{n-1}, \ldots, a_1)]}$ and
$\mathbf{B=[\Sigma_{i:\, m\le p\ldots1}\, y_i/v_i,\ (k_B, s_B, b_m, b_{m-1}, \ldots, b_1)]} \in$ **S(U)**,
the binary operation $\Diamond$ on **S(U)**x**S(U)** is defined as follows:

$$\mathbf{A \Diamond B = [\Sigma_{i:n+m-1\ldots1}\, z_i/w_i,\ (k_{A+}k_B, 1, c_{n+m-1}, \ldots, c_1)]}$$

where

$$\mathbf{w_i = \bigcup_{\substack{h=1\ldots i\\ k=i\ldots1\\ h\le n, k\le m}} (u_h \cap v_k}$$

fuzzy parts

$$z_i = \frac{s_A s_B}{(k_A + k_B)\, c_i} \sum_{\substack{h = 1\ldots i \\ k = i\ldots 1 \\ h \le n,\, k \le m}} a_h b_k (k_A x_h + k_B y_k$$

$$c_i = \sum_{\substack{h = 1\ldots i \\ k = i\ldots 1 \\ h \le n,\, k \le m}} a_h b_k$$

crisp parts

It is worth noting that A ◊ **0** = **0** and A ◊ **1** = A.

The indices a_h e b_k represent the number of sets that have generated the i-th class of A and B, respectively. The indices k_A e k_B represent, in turn, the number of sets that have generated the classes of A and B, respectively. The operation for z_i represents essentially a mean among the type-2 fuzzy sets, where each fuzzy set takes a weight in some way related to the changes induced by the composition. Essentially these indices include the computational history of the type-2 fuzzy sets. The operation ◊ is well defined: *i*) $(w_{n+m-1}, w_{n+m-2} \ldots, w_1) \in$ ***P***(**U**); *ii*) the t-uple $(c_{m+n-1}, \ldots\ldots, c_1)$ is strictly increasing and symmetric with respect to the central values; *iii)* A◊B∈**S**(U); *iv*) the elements z_i are triangular fuzzy numbers on [0, 1].

Proposition 1: The structure (**S(U)**, ◊, **1**) is a commutative monoid.
Proof. [10] ♦

Example 1: Consider the universe U={a, b, c, d, e} and the following elements in **S(U)**:
A = [[0.8, 1, 1]/a +[0.5, 0.7, 0.9] / {b, e}+[0, 0, 0.2]/{c, d}, (1,1,1,1,1)]
B = [[0.5, 0.7, 0.9]/{d, a} + [0.2, 0.4, 0.6]/{b, c, e}, (1,1,1,1,1)]

Let x_1 = [0, 0, 0.2], x_2 = [0.5, 0.7, 0.9], x_3 = [0.8, 1, 1], y_1 = [0.2, 0.4, 0.6], y_2 = [0.5, 0.7, 0.9], $a_3=a_2=a_1= b_3=b_2=b_1=1$, $k_A=k_B=1$.

The computation of the crisp parts for A◊B is carried out as follows:

	{a}	{b, e} {a, d}	{c, d} {b, c, e}
{a, d}∩{a}	{b, c, e}∩{a} ∪ {a, d}∩{b, e}	{b, c, e}∩{b, e} ∪ {a, d}∩{c, d}	{b, c, e}∩{c, d}
{a}	∅	{b, d, e}	{c}
w_4	w_3	w_2	w_1

The computation of the coefficients c_i can be organized as follows:

$c_1 = a_1b_1 = 1$; $c_2 = a_1b_2 + a_2b_1 = 2$
$c_3 = a_1b_3 + a_2b_2 + a_3b_1 = 2$, since $b_3 = 0$
$c_4 = a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1 = 1$, since $b_4 = a_4 = 0$
$k_c = k_A + k_B = 2$, $s_c = 1$

The elements z_i (fuzzy parts) are singled out as follows:

$z_1 = ((s_A*s_B)/(k_c*c_1))*(a_1b_1*(k_Ax_1+ k_By_1)) = (1*1)*/(1*1)) * (1*1*(1*[0.2, 0.4, 0.8]+1*[0.2, 0.4, 0.6]) = [0.1, 0.2, 0.4]$

$z_2=((s_A*s_B)/(k_c*c_2)*((a_1b_2*(k_Ax_1+ k_By_2) + a_2b_1*(k_Ax_2+ k_By_1)) = ((1*1)/(2*2)) * (1*1*(1*[0.0, 0.0, 0.2] + 1*[0.5, 0.7, 0.9]) + +1*1*(1* [0.5, 0.7, 1.1] + 1*[0.2, 0.4, 0.6]) = [0.3, 0.45, 0.65]$

$z_3 = ((s_A*s_B) / (k_c*c_3))*((a_3b_1*(k_Ax_3+ k_By_1) + a_2b_2*(k_Ax_2+ k_By_2)) = [0.5, 0.7, 0.85]$

$z_4 = ((s_A*s_B) / (k_c*c_4))*((a_3b_2*(k_Ax_3+ k_By_2)) =[0.65, 0.0.85, 0.95]$.

So we obtain C= A ◊ B=

[[0.65, 0.85, 0.95] /a + [0.5, 0.7, 0.85] / {∅} +
[0.3, 0.45, 0.65]/{b, d, e} + [0.1, 0.2, 0.4]/{c}, (2, 1, 1, 2, 2, 1)].

4. THE ORDER RELATION

An order relation is now introduced on **S(U)**:

$$A \leq B \text{ iff exists } C \in S(U) \text{ such that } A=B \lozenge C.$$

The element C is denoted by B ↓ A. In particular one has **0** ↓ **0** = **1**. If neither Á≤B nor B≤A, one writes **A** || **B**. The algorithm for calculating C such that A = B ◊ C is given in [7, 8], where the following proposition is proved:

Proposition 2: The solution C of the equation A=B◊C is C=B↓A iff exist B↓A and B◊(B↓A) = A. ♦

The properties of the binary relation are proved as follows:

i) *Reflexivity*: A≤A since A=A◊**1**;
ii) *Antisymmetricity*: if A≤B and B≤A then A= B ◊ C and B =A◊ C' for some C and C', then A=A ◊ C ◊ C'. It follows C ◊ C' = **1**, since A ↓ A = **1**, for any A ∈ **S(U)**. Since X ∈ **S(U)** such that A ◊ X = **1** does not exist, from C ◊ C' = **1** follows C = C' = **1**. Finally A = B ◊ **1** = B;
iii) *Transitivity*: if A≤B and B≤C then A= B ◊ D and B = C ◊ E for some D and E. Then A= C◊E◊D, so A≤C.

It is immediate to verify that **1** is the top element of **S(U)** and **0** is the bottom. From the definition the following results are obtained: A↓ A = **1**, **1**↓A = A, A↓ **0** = **0**. Moreover neither **0**↓A nor A↓**1** are not defined, if A≠**1** and A≠**0**, respectively.

The basic properties of the order relation and the operation ↓ are as follows:

P_0: A◊B ≤ A;
P_1: If A◊C = B◊C and C ≠**0**, then A=B;
P_2: i) If A ≤ B then A◊C ≤ B◊C; ii) If A◊C ≤ B◊C and C ≠ **0**, then A ≤ B;
P_3: A ≤ B iff A ≤ B↓A;
P_4: A ≤ B iff (B↓A)↓A= B;
P_5: If A ≤ B ≤ C≠0 then C↓A ≤ C↓B; (*isotone in the right element*)
P_6: If C ≤ A ≤ B then B↓C ≤ A↓C; (*antitone in the left element*)
P_7: A↓(A◊B) = B
P_8: If C ≤ A then (A◊B)↓(B◊C) = A↓C
P_9: If C ≤ B and C ≤ A◊B then A↓ (B↓C) = (A◊B)↓C

The proofs are trivial, for example: P_9: If C ≤ B and C ≤ A◊B then there are K and H∈**S(U)** such that C=B◊K and C=A◊B◊H, so we have K=A◊H, B↓C=A◊((A◊B)↓C) and A↓(B↓C) = (A◊B)↓C). ♦

Thus one obtains:
Proposition 3: (**S(U)**, ≤, ◊, **1**)) is a partially ordered commutative monoid.
Proof. If A≤B and C≤D, then there are H, K ∈ **S(U)** such that A=B◊H and C=D◊K. One gets: A◊C= B◊D◊H◊K ≤ B◊D, by P_0. ♦

Example 2: Let U = {a, b, c, d, e, f, g, h, e} be a universe of discourse and be the following type-2 fuzzy sets in **S(U)**:
A = [[0.8 1 1]/{a, b} + [0.5 0.7 0.9]/{c, d, e} [0.2 0.4 0.6]/{f, g}+ [0.0 0.0 0.2]/{h}, (1, 1, 1, 1, 1, 1)];
C = [[0.8 1 1]/{b} + [0.5 0.8 0.9]/{a} [0.4 0.6 0.7]/{c} + [0.2 0.4 0.6]/{e}+ [0.1 0.2 0.4]/{d, f, g}[0.0 0.0 0.4]/{h}, (2, 1, 1, 2, 3, 3, 2,1)]
In [8] the solution X of the A◊X = C in **S(U)** is found. One gets:
X = [[0.8, 1.0, 1.0]/{f,g,h,e} + [0.2, 0.4, 0.6]/{a,c,d} +
[0.0, 0.0, 0.2]/{b}, (1, 1, 1,1,1)].
It is easy to verify that A◊X = C. One can adfirm X = A↓C and A◊ (A↓C) = C, hence C≤A.

5. THE RESIDUATED MONOID

Now the operation A↓B is extended so that the new operation is the residuum of the operation ◊ in the monoid **S(U)**.
Definition:

$$\mathbf{A \to B} = \begin{cases} \mathbf{A \downarrow B} & \mathbf{if} \quad \mathbf{B \le A} \\ \mathbf{1} & \mathbf{if} \quad \mathbf{A \le B} \\ \mathbf{B} & \mathbf{if} \quad \mathbf{A \,||\, B} \end{cases}$$

Proposition 4:
- **i)** If B≤C then A → B ≤ A → C;
- **ii)** If B≤C then C → A ≤ B → A,
- **iii)** B ≤ A → A◊B,
- **iv)** A◊(A → B) ≤ B.

Proof.

(i) Let us consider three cases. *Case 1a*: B≤C≤A. We have A → B = A ↓ B ≤ A ↓ C = A → C, by property P_5. *Case 1b*: B≤A≤C. One gets: A → B = A ↓ B ≤ **1** = A → C. *Case 2*: A≤B≤C. One has: A → B = **1** = B → C. *Case 3*: A | | B. In such case it is A | | C. We have A→B=B≤C=A→C.

(ii) The proof is done by enumerating three cases. *Case 1a*: B≤C≤A. One has: C → A = **1** = B → A. *Case 1b*: B≤A≤C. One gets: C → A = C ↓ A ≤ **1** = B → A. *Case 2*: A≤B≤C. One gets: C → A = C ↓ A ≤ B ↓ A, by properties P_6. *Case 3*: A | | B. In such case it is A | | C. One has: C → A = A = B → A.

(iii) From A◊B ≤ A we derive A → A◊B = A↓(A◊B) = B by property P_7.

(iv) If A≤B then A◊(A → B) = A◊1= A; if B≤A then A◊(A → B)=A◊(A↓B)=B by definition; if A | | B then A◊(A → B) = A◊B ≤B by property P_0.♦

The pair (◊, →) satisfies the Galois relation, as is shown in the following:

Proposition 5: C◊A ≤ B iff C ≤ A → B.

Proof. **(⇒)** Case 1: B≤A. We have C◊A ≤ C ≤ A → A◊C ≤ A → B, from the previous proposition 5(iii). Case 2: A≤B. One gets: C◊A≤B=**1**=A→B. Case 3: A | | B. One has: C◊A≤B=A→B. **(⇐)** C ≤ A→B then C◊A≤A◊(A → B) (=A, if A ≤ B); (=A◊(A↓B) = B, if B≤A); (=A◊B≤B, if A | | B).♦

Hence:

Proposition 6: (**S(U)**, ◊, →, **1**) is a residuated monoid. ♦

6. A TOTALLY ORDERED BL-ALGEBRA

Let us consider the following example. Let U be a set of individuals who are to be evaluated with respect to heart failure risk and cholesterol levels by means of suitable type-2 fuzzy sets A and B, respectively. It is known that the metabolic syndrome increases the risk of heart disease. Using this algebra, one says that there is another type-2 fuzzy set C such that A = B◊C, where C denotes the other factors responsible for heart disease. This implies that A≤B, namely A and B are ordered, belonging to the same chain in **S(U)**. In many applications one has that the properties {P_1, P_2, ..., P_n} of the elements of the universe of discourse are *related* among them and this relation is treated in this algebra so that the set of properties is a linearly ordered set. In the present section the algebraic properties of the chains of **S(U)** are studied.

Let $\mathbf{S_C(U)}$ a chain of **S(U)**. By $\mathbf{S_C(U)}$ one understands a subset of totally ordered elements of **S(U)** under the previous ordering relation, containing **0** and **1**.

Any chain $\mathbf{S_C(U)}$ is a lattice by the usual operations:

$$A \wedge B = \min(A, B),\ A \vee B = \max(A, B).$$

Moreover one defines:

$$A \to_C B = \begin{cases} A \downarrow B & \text{if} \quad B \le A \\ 1 & \text{if} \quad A \le B \end{cases}$$

So it is easily proved:

Proposition 7: In any **Sc(U)**: i) A ◊ C ≤ B iff C ≤ A $\to_c$ B; ii) A $\to_C$ B = $\vee_{C \in \mathbf{Sc(U)}}$ {C: A ◊ C ≤B }

The two sentences are equivalent, as it is well known. They both claim that the couple (◊, $\to_c$) is an *adjoint pair*, thus the first main result is the following:

Proposition 8: Any chain ($\mathbf{S_C(U)}$, $\wedge$, $\vee$, $\Diamond$, $\to_C$, **1, 0**) is a totally ordered BL-algebra.
Proof. It is enough to verify that $(A \to_C B) \vee (B \to_C A) = \mathbf{1}$, and this is easily proved. ♦

It is worth recalling this important result [17; lemma 2, p.870]: For a totally ordered BL-algebra **BL** the following are equivalent:

1. **BL** is sum irriducible;
2. $A \to B = B$ iff either A=**1** or B=**1**;
3. **BL** is a Wajsberg algebra.

In [17] the reader will find the definitions of Wajsberg algebra and sum irreducible of a BL-algebra. Moreover these results hold true:
i) a Wajsberg algebra is a bounded Wajsberg hoop;
ii) a Wajsberg hoop is a hoop satisfying the equation $(A \to B) \to B = (B \to A) \to A$;
iii) a hoop is an algebra (S, $\to$, *, e) such that (S, *, e) is a commutative monoid and for any A, B, C $\in$ S:

1. $A \to A = \mathbf{1}$;
2. $A*(A \to B) = B*(A \to B)$;
3. $A \to (A \to C) = A*B \to C$.

Finally, BL-algebras are particular bounded hoops [1, 17]. Hence the algebra ($\mathbf{S_C(U)}$, $\Diamond$, $\to_C$, **1**) is a bounded hoop.

7. Ordinal Sums of Type-2 Fuzzy Sets BL-Algebras

It is now shown how to compose different BL-chain to get new BL-algebras, in particular BL-chains. The second main result is the following:

Proposition 9: Let I a totally ordered set with minimum i_0. Let ($\mathbf{S_Ci(U)}$, $\wedge_i$, $\vee_i$, $\Diamond_i$, $\to_{Ci}$, **1, 0**), be a family a BL-chains. Then

$$\mathbf{BLC} = (\cup_{i\in I}\mathbf{S_C i(U)}, \wedge_e, \vee_e, \Diamond_e, \to_e, \mathbf{1}, \mathbf{0})$$

where:

BLC1.1:

$$A \wedge_e B = B \wedge_e A = \begin{cases} \min(A,B) & \text{if } A, B \in S_{Ci}(U) \\ A & \text{if } A \in C_i, B \in C_j, i<j \end{cases}$$

BLC1.2:

$$A \vee_e B = B \vee_e A = \begin{cases} \max(A,B) & \text{if } A, B \in S_{Ci}(U) \\ B & \text{if } A \in C_i, B \in C_j, j>i \end{cases}$$

BLC2:

$$A \Diamond_e B = B \Diamond_e A = \begin{cases} A \Diamond B & \text{if } A, B \in S_{Ci}(U) \\ A & \text{if } A \in C_i, B \in C_j, j>i \end{cases}$$

BLC3:

$$A \to B = \begin{cases} A \to_{C_i} B & \text{if } A, B \in S_{Ci}(U) \\ 1 & \text{if } A \in S_{Ci}(U),\ B \in S_{Cj}(U), i<j \\ B & \text{if } A \in S_{Ci}(U),\ B \in S_{Cj}(U), j<i \end{cases}$$

is a **BL-chain**.
Proof: The *operations BLC1.1* and *BLC1.2* induce lattice properties to **BLC**. If A and B belong to same chain, the operation reduces to min and max operations, previously discussed. The remaining cases are treated as min and max

between indices in I. So the lattice properties i)...iv), given in section II, can be easily proved by straightforward computation, with $* = \Diamond_{\mathbf{e}}, \cap = \wedge_e, \cup = \vee_e$.

The *total order* $\leq_e$ is defined in the usual way:

$$A \leq_e B \text{ iff } A \wedge_e B = A$$

The *operation* BLC2 $\Diamond_{\mathbf{e}}$, extends the monoidal properties of **S(U)** to $\cup_{i\in I}\mathbf{S_C i(U)}$. To verify that the algebraic features are preserved is straightforward. If $\Diamond_{\mathbf{e}}$ reduces to $\Diamond$ the proof is similar to the previous one. In the other cases it is enough to consider that, with $A\in\mathbf{S_{Ci}(U)}$, $B\in \mathbf{S_{Ci}(U)}$, if $i<j$ we have $C\Diamond_{\mathbf{e}}A= C\Diamond_{\mathbf{e}}A\leq_{\mathbf{e}}A=B\Diamond_{\mathbf{e}}A$; if $j<i$ we have $C\Diamond_{\mathbf{e}}A=C\Diamond_{\mathbf{e}}A\leq B=B\Diamond_{\mathbf{e}}A$. The property P_2 is used.

This fundamental relation holds true:

$$\text{If } C \leq_e B \text{ then } C\Diamond_{\mathbf{e}}A\leq_e B\Diamond_{\mathbf{e}}A$$

The *implication* BLC3 is the residuum of $\Diamond_{\mathbf{e}}$, as shows the following:

Proposition 10: For any A, B, C $\in$ **BLC**,

$$C\Diamond_{\mathbf{e}}A \leq_e B \text{ iff } C \leq_e A \rightarrow_e B$$

Proof. Consider only the interesting cases when A, B, C belong to different chains. **(⇒)** If $A\in\mathbf{S_{Ci}(U)}$, $B\in\mathbf{S_{Cj}(U)}$, then one has: if j>i then $C \Diamond_{\mathbf{e}} A \leq_e B \leq_e \mathbf{1}=A \rightarrow_e B$; if i>j then $C\Diamond_{\mathbf{e}} A \leq_e B = A \rightarrow_e B$. **(⇐)** If $C\leq_e A\rightarrow_e B$ then $C\Diamond_e A\leq_e A\Diamond_e(A \rightarrow_e B) =$ $(= C\Diamond_e A \leq_e A\leq_e B$, if $A \leq_e B)$; $(=C\Diamond_{\mathbf{e}} A\leq_e A\Diamond_e(A\downarrow B)$, if $B \leq_e A)$; $(= C \Diamond_e A\leq A \rightarrow_e B = B$, if $A \mid\mid B)$.♦

The residuum operation $\rightarrow_e$ satisfies the usual general properties:

R_1: $\mathbf{1}\rightarrow_e A = A$
R_2: $A \Diamond_{\mathbf{e}} (A \rightarrow_{\mathbf{e}} B) \vee_e B = B$
R_3: $(A \rightarrow_e A \Diamond_{\mathbf{e}} B) \wedge_e B = B$
R_4: $B \Diamond_{\mathbf{e}} (B \rightarrow_e A) = (B \rightarrow_e A) \Diamond_{\mathbf{e}} [(B \rightarrow_{\mathbf{e}} A) \rightarrow_{\mathbf{e}} A]$
R_5: $A \Diamond_{\mathbf{e}} (B \vee_e C) = (A\Diamond_e B) \vee_e (A \Diamond e\, C)$
R_6: $A \rightarrow_e (B \rightarrow_e C) = A \Diamond_{\mathbf{e}}\ B \rightarrow_e C$
R_7: $A \rightarrow_e (B \wedge_e C) = (A \rightarrow_{\mathbf{e}} B) \wedge_e (A \rightarrow_e C)$

The following proposition ends the proof that BLC is a totally ordered BL-algebra:

Proposition 11: For any A, B, C $\in$ **BLC**,
i) $A \wedge_{\mathbf{e}} B = A\Diamond_{\mathbf{e}} (A \rightarrow_e B)$
ii) $(A \rightarrow_e B) \cup (B \rightarrow_e A)$

Proof. When $\wedge_{\mathbf{e}} = \wedge$ and $\rightarrow_e = \rightarrow_c$ no problem. Consider only the cases when A and B belong to different chains. i): Suppose that $A\in\mathbf{S_{Ci}(U)}$ and $B\in\mathbf{S_{Cj}(U)}$, then one has: if j>i then $A\Diamond_{\mathbf{e}}(A\rightarrow_e B) = A\Diamond_{\mathbf{e}}\mathbf{1} = A = A\wedge_{\mathbf{e}} B$; if i>j then $A\Diamond_{\mathbf{e}}(A\rightarrow_e B) = =A\Diamond_{\mathbf{e}}B = B = A\wedge_{\mathbf{e}} B$. ii) Proved in a similar way.♦

Finally: **BLC** is a *lattice* (BLC1.1, BLC1.2), an ordered *commutative monoid* (BLC2), it has a *Galois adjoint pair* (BLC3), satisfies the condition of *pre-linearity* (Proposition 11, ii)) and the *relation* between the lattice operation $\wedge_{\mathbf{e}}$, the monoidal operation $\Diamond_{\mathbf{e}}$, and his residuum $\rightarrow_e$. (Proposition 11, i)), therefore is a BL-algebra, more specifically a BL-chain. ♦

This BL-chain is denoted as the *ordinal sum* of the family $\{ \mathbf{S_{Ci}\ (U)} \}_{i\in I}$.[17]

8. APPLICATIONS

This algebraic approach has been applied to different fields that can be summarized as follows:

In [5] a method for environmental evaluation is illustrated based on type-2 fuzzy sets and an algebra defined therein. In the evaluation model every environmental indicator (security, facilities, environment, social impact, and so on) is a type-2 fuzzy set. The model has been utilized both to rank the quality of life in some Italian cities and then the environmental quality of four sites for a domestic airport near Reykjavik. As regards the Italian cities our approach produces results similar to those obtained by statistical methods but one gets a linguistic classification of cities and not a numerical one. Also as regards the sites for the airport, the results are similar but the linguistic expressivity is greatly enhanced.

The paper [6] shows the behaviour of a commutative l-monoid, endowed with a suitable operation of composition, as regards the problem of classification with fuzzy attributes. The concepts of relevance and similarity are introduced, then a mechanism for weighing the attributes is shown. Finally, a case study concerning graphology is illustrated in details.

In [7] the representation and management of a fuzzy hypermedia is formalized introducing linguistic resource and psycho-cognitive profile of the user. The model uses a correspondence between the content of information in the hypermedia domain and representation of the users. The stereotype users and the real users are represented by type-2 fuzzy sets. The domain of the latter is the set of nodes of the hypertext whereas the range are the terms of a t-uple of linguistic variables used in the definition of the users's features. The user model dynamically updates the representation of the real users, realizing full adaptivity with respect both to the presentation and to the navigation.

In [8] a model of differential medical diagnosis is illustrated, based on type 2 fuzzy sets. Given the patient history case and a set of suspected-inherent pathologies, the method creates a set of linguistically labelled clusters of the pathologies. The pathologies present in each cluster are ordered through a similarity function. In the case of lack of information, the method indicates the elements needed to have a more precise diagnosis and it can control its same accuracy

The paper [9] presents a model for the fuzzy-based analysis of diabetic neuropathy, whose pathogenesis so far is not well known. The underlying algebraic structure is a commutative l-monoid, whose support is a set of classifications based on the concept of linguistic variable. The analysis is carried out by means of patient's anagraphical and clinical data, e.g. age, sex, duration of the disease, insulinic needs, severity of diabetes, possible presence of complications. The results obtained are identical with medical diagnoses. Moreover, analyzing suitable relevance factors one gets reasonable information about the etiology of the disease.

In [10] classifications are obtained through clusters composed of conventional sets and fuzzy attributes. The expressive power of the method is such that several situations can be viewed as classification problems, e.g., fuzzy assessment of students, user modelling for fuzzy hypermedia systems, spaces of the cognitive states of the user of a tutoring system, financial investments, medical diagnoses. The problem of getting the unknown classification starting from the final classification is investigated and it is shown that the problem is strictly related to the solution of an equation in the monoid. Finally, by means of this approach, both the absolute and the relative relevance of an attribute are defined and evaluated, given a universe of discourse and a set of classifications.

9. CONCLUSIONS

In this paper the features of a peculiar BL-algebra, defined on some lattices of a specific class of type-2 fuzzy sets, are presented. This algebra has been used in several application fields, using suitable functions, that can be summarized as follows: **i)** *The linguistic approximation*, which allows to build new linguistic terms from triangular fuzzy numbers obtained by composition in **S(U)**, using some basic linguistic term and linguistic hedges; **ii)** *Index of resemblance* between two elements of **S(U)**; **iii)** *Method for modifying type-2 fuzzy sets*, which allows to modify elements of **S(U)** considered more relevant than others with respect to the specific data analysis to be carried out; **iv)** *The relevance function*, which measures the ability of A (B, respectively) in influencing B (A, respectively) in order to get C=A◊B.

We think that the properties of this algebra deserve further investigation. In particular, we believe that the t-norms properties of the monoidal operation could be used for defining a logical language and a related semantics. Another goal is to endow this algebra with type-2 fuzzy similarity and a suitable homomorphism.

As regards application fields, two areas are currently under investigation:

Document Search: introduction of linguistic terms to enrich the document metadata and represent user profiles. An algorithm for matching user profiles and document metadata could cluster and order the results depending on user needs.

Learning Assessment: construction of an adaptive automatic system that selects the most suitable questions for evaluating the student taking into account the sequence of previous learning steps.

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