A Generalization of M-Series and Integral Operator Associated with Fractional Calculus

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ABSTRACT--- ***This paper is devoted for the study of a new generalization of M-series. Its various properties including differentiation, recurrence relation, Laplace transform, Beta transform, Mellin transform, Generalized hypergeometric series form, Mellin Barnes integral representationand its relationship with Fox's H-function and Wright hypergeometric function are investigated and established. The integral operator*** $\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta}{\mathcal{M}}}$ ***containing the generalized M-Series*** $\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}}(z)$ ***in the kernel is defined and studied namely its boundedness on*** $L(a,b)$***. Also composition of Riemann – Liouville fractional integration and differentiation, and Hilfer's fractional derivative operator with*** $\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta}{\mathcal{M}}}$ ***are established.***

Keywords--- GeneralizedM-Series, H-function , Integral transforms,Fractional Integral and Differential Operators.

1. INTRODUCTION

In 2008 the mathematicianManoj Sharma [14] introduced the M-series

$$ {}_p\overset{\alpha}{M}_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;z\right) = {}_p\overset{\alpha}{M}_q(z) $$

$$ {}_p\overset{\alpha}{M}_q(z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{\Gamma(\alpha k+1)} \tag{1.1} $$

Where $z,\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha)>0$ *and* $(a_j)_k, (b_j)_k$ are the Pochammer symbols.

The series in (1.1) is defined when none of the parameters $b_j;s;\, j=1,2,\ldots,q$ is a negative integer or zero, If any numerator parameter a_j is a negative integer or zero, then the series terminates to a polynomial of z . From the ratio test it is evident the series in (1.1) is convergent for all z if $p \le q$, also if $p=q+1$ its convergent absolutely or conditionally when $|z|=1$, and divergent if $p>q+1$.

The series in (1.1) is a particular case of the $\bar{H}$ - function of Inayat – Hussain [3]. The M-series is interesting because the ${}_pF_q$ - hypergeometric function and the Mittag – Leffler functions [1,11] follow as its particular cases, and these functions have recently found essential applications in solving problems in physics, engineering and applied sciences.
Further extension of both Mittag – Leffler function and generalized hypergeometric function ${}_pF_q$ is called generalized M-series introduced and studied by Sharma and gain [15] where ,

$$ \underset{p,q}{\overset{\alpha,\beta}{M}}(z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{\Gamma(\alpha k+\beta)} \tag{1.2} $$

The series in (1.2) is convergent for all z if $p \le q+\mathrm{Re}(\alpha)$, also it is convergent for $|z|<\delta=\alpha^{\alpha}$ if $p=q+\mathrm{Re}(\alpha)$ and divergent if $p>q+\mathrm{Re}(\alpha)$.

On the other hand many authors stated and proved interesting examples of the special functions of fractional calculus [6,9] (SF of FC), a notion that gained recently an important role in the theory of differentiation of arbitrary order and in the solution of fractional order differential equations.

Salim and Faraj [11] defined a fractional integral operator $\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,w,c^+}(z)$ containing the generalized Mittag-Leffler function $E^{\gamma,\delta,q}_{\alpha,\beta,p}(z)$ in its kernel , they studied that operator on $L(a,b)$ space of Lebesgue measurable functions namely its boundedness.

In continuation of studying special functions of fractional calculus Sharma and gain [15] gave representation of the generalized M-series $\underset{p,q}{\overset{\alpha,\beta}{M}}(z)$ with formulas of fractionalcalculus operators

In this paper a new generalization of M-series introduced by the authors as,

$$\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}}\left(a_1,\dots,a_p;b_1,\dots,b_q;z\right):=\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}}(z)$$

$$\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}}(z)=\sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{z^k}{\Gamma(\alpha k+\beta)} \qquad (1.3)$$

where $z,\alpha,\beta\in C,\ \ \mathrm{Re}(\alpha)>0\ \ and\ \ m,n$ are non-negative real number (1.4)

The conditions of convergence of the series (1.3) is discussed and established, also all possible special cases of the M-series (1.3) are stated.Recurrence relations, Derivation formulas, Laplace transform, Beta transform, Mellin transform, MellinBarnes integral representation of $\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}}(z)$ are established, also its relationship to Fox's H-function and Wrighthypergeometric function is under concentration.

The integral operator defined by

$$\left(\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta}{\mathcal{M}}}\varphi\right)(x)=\int_c^x(x-t)^{\beta-1}\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}}\left[w\,(x-t)^{\alpha}\right]\varphi(t)dt \qquad (1.5)$$

containing the generalized M-series (1.3) in its kernel is investigated and its boundedness is proved under certain conditions.

Theorems of composition of fractional calculus operators,

$$\left(I^{\lambda}_{c^+}\varphi\right)(x)=\frac{1}{\Gamma(\lambda)}\int_c^x(x-t)^{\lambda-1}\varphi(t)dt \qquad \left(\lambda\in\mathbb{C},\ \mathrm{Re}(\lambda)>0\right) \qquad (1.6)$$

and

$$\left(D^{\lambda}_{c^+}\varphi\right)(x)=\left(\frac{d}{dx}\right)^s\left(I^{s-\lambda}_{c^+}\varphi\right)(x) \qquad s=\left[\mathrm{Re}(\lambda)\right]+1 \qquad (1.7)$$

with integral operator defined in (1.5) are given and proved.

As a matter of fact if $w=0$, $m=1$ and $n=1$, then the integral operator (1.5) corresponds essentially to the Riemann-Liouville fractional integral operator defined in(1.6).

The generalized fractional derivative operator $D^{u,v}_{\lambda}\varphi$ known as Hilfer's derivative [2] is written as,

$$\left(D^{u,v}_{c^+}\varphi\right)(x)=\left(I^{v(1-u)}_{c}\frac{d}{dx}\left(I^{(1-v)(1-u)}_{c^+}\varphi\right)\right)(x) \qquad (1.8)$$

$D^{u,v}_{c^+}$ yields the classical Riemann-Liouville fractional derivative $D^{u}_{c^+}$ when $v=0$, also if $v=1$ it reduces to Caputo fractional derivative.

Throughout this paper, we need the following well-known facts and rules

- Beta transform (Sneddon [17])

$$B\left\{f(z);a,b\right\}=\int_0^1 z^{a-1}(1-z)^{b-1}f(z)dz \qquad \mathrm{Re}(a)>0,\ \mathrm{Re}(b)>0 \qquad (1.9)$$

• Laplace transform (Sneddon [17])

$$\mathcal{L}\{f(z);s\} = \int_0^\infty e^{-sz} f(z)dz \qquad \operatorname{Re}(s) > 0 \qquad (1.10)$$

and

$$\mathcal{L}\left\{\frac{t^{n-1}}{\Gamma(n)};s\right\} = \frac{1}{s^n} \qquad n > 0 \qquad (1.11)$$

•Mellin transform (Sneddon [17])

$$\mathcal{M}\{f(x);s\} = f^*(s) = \int_0^\infty z^{s-1} f(z)dz \qquad n > 0 \qquad (1.12)$$

and the inverse Mellin transform is given by

$$f(z) = \mathcal{M}^{-1}\{f^*(s);z\} = \frac{1}{2\pi i} = \int_{c-i\infty}^{c+i\infty} z^{-s} f^*(s)ds \qquad c \in \mathbb{R} \qquad (1.13)$$

•Wright generalized hypergeometric function (Srivaslava and Manocha [18])

$${}_p\Psi_q\left[\begin{matrix}(a_1,A_1)\ldots(a_p,A_p)\\(b_1,B_1)\ldots(b_q,B_q)\end{matrix};z\right] = \sum_{n=0}^\infty \frac{\prod_{i=1}^p \Gamma(a_i + A_i n)}{\prod_{j=1}^q \Gamma(b_j + B_j n)} \frac{z^n}{n!} \qquad (1.14)$$

•Fox's H-function (Kilbas and Saigo [4])

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^m \Gamma(b_i + \beta_i s)\prod_{j=1}^n \Gamma(1-a_i-\alpha_i s)}{\prod_{j=m+1}^q \Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^p \Gamma(a_j+\alpha_j s)} \qquad (1.15)$$

• Generalized hypergeometric function (Rainville [10])

$${}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z) = \sum_{n=0}^\infty \frac{\prod_{i=1}^p \Gamma(\alpha_i)_n}{\prod_{j=1}^q \Gamma(\beta_j)_n} \frac{z^n}{n!} \qquad (1.16)$$

•Fubini's theorem (Dirichlet formula) [12]

$$\int_a^b dx \int_a^x f(x,t)dt = \int_a^b dt \int_t^b f(x,t)dx \qquad (1.17)$$

$$\frac{d}{dx}\int_a^x h(x,t)dt = \left[\int_a^x \frac{\partial}{\partial x} h(x,t)dt\right] + h(x,x) \qquad (1.18)$$

• Caputo fractional derivative [12]

$$\left(C_c^\lambda \varphi\right)(x) = I_{c^+}^{n-\lambda} \frac{d^n}{dx^n}\varphi(x) \qquad (1.19)$$

• $L(a,b)$ Space of Lebesgue measurable function on $[a,b]$

$$L(a,b) = \left\{g(x) : \|g\|_1 = \int_a^b |g(x)|dx < \infty\right\} \qquad (1.20)$$

Also we need the following relations [10]

- $$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \tag{1.21}$$
- $$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \tag{1.22}$$
- $$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{1.23}$$
- $$(\alpha)_{n+k} = (\alpha)_n (\alpha+n)_k \tag{1.24}$$
- $$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} \tag{1.25}$$

2. BASIC PROPERTIES

Some special cases of the generalized M-series $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)$ are the following

(i) When $m = n = 1$, in (3.1) we get the generalized M-series introduced by Sharma and gain [15]

$$\underset{p,q;1,1}{\overset{\alpha,\beta}{M}}(z) = \underset{p,q}{\overset{\alpha,\beta}{M}}(z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{\Gamma(\alpha k+\beta)} \tag{2.1}$$

(ii) For $m = n = 1$ and $\beta = 1$ in (3.1) the result will be the M-series defined by Sharma [14]

$$\underset{p,q;1,1}{\overset{\alpha,1}{M}}(z) = \underset{p,q}{\overset{\alpha}{M}}(z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{\Gamma(\alpha k+1)} \tag{2.2}$$

(iii) The generalized Mittag – Leffler function [11] introduced by Salim and Farajyields when $p = q = 1$ in(3.1) , For more result see also [11,16]

$$\underset{1,1;m,n}{\overset{\alpha,\beta}{M}}(z) = E_{\alpha,\beta,n}^{a_1,b_1,m}(z) = \sum_{k=0}^{\infty} \frac{(a_1)_{km}}{(b_1)_{kn}} \frac{z^k}{\Gamma(\alpha k+\beta)} \tag{2.3}$$

(iv) For $p = 0,\ q = 1,\ n = 1$ and $b_1 = 1$ in (3.1) , the Wrighthypergeometric function $W(z;\alpha,\beta)$ [6] comes

$$\underset{0,1;-,1}{\overset{\alpha,\beta}{M}}(-;1;z) = {}_0\Psi_1\left[\begin{matrix} - \\ (\beta,\alpha) \end{matrix} ;\ z\right] \tag{2.4}$$

(v) The generalized hypergeometric function ${}_pF_q(z)$ [8] when $\alpha = \beta = 1$ and $m = n = 1$, with arbitrary p,q we have

$$\underset{p,q;1,1}{\overset{1,1}{M}}(z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!} = {}_pF_q\left((a_i)_1^p;(b_j)_1^q;z\right) \tag{2.5}$$

Theorem 2.1:

The series in (1.3) in absolutely convergent for all values of z provided that $pm < qn + \operatorname{Re}(\alpha)$, moreover if $pm = qn + \operatorname{Re}(\alpha)$ the series converges for $|z| < \delta = \alpha^{\alpha}$

Proof:

Rewriting $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)$ in the form of power series $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z) = \sum_{k=0}^{\infty} d_k z^k$

when $d_k = \dfrac{(a_1)_{km} \cdots (a_p)_{km}}{(b_1)_{kn} \cdots (b_q)_{kn} \Gamma(\alpha k+\beta)}$ and applying

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1 + \frac{(a-b)(a+b-1)}{2z} + O\left(\frac{1}{z^3}\right)\right]$$,see [19]

we get,

$$\left|\frac{c_{k+1}}{c_k}\right| = \left|\frac{(a_1)_{km+m}}{(a_1)_{km}}\cdots\frac{(a_p)_{km+m}}{(a_p)_{km}}\cdot\frac{(b_1)_{kn}}{(b_1)_{kn+n}}\cdots\frac{(b_q)_{kn}}{(b_q)_{kn+n}}\cdot\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha k+\beta+\alpha)}|z|\right|$$

$$= \left[(mk)^m\left[1+\frac{m(2a_1+m)}{2mk}\right]+O\left(\frac{1}{(mk)^3}\right)\cdots(mk)^m\left[1+\frac{m(2a_p+m)}{2mk}\right]+O\left(\frac{1}{(mk)^3}\right)\right.$$

$$.(nk)^{-n}\left[1+\frac{-n(2b_1+n)}{2nk}\right]+O\left(\frac{1}{(nk)^3}\right)\cdots(nk)^{-n}\left[1+\frac{-n(2b_q+n)}{2nk}\right]+O\left(\frac{1}{(nk)^3}\right)$$

$$\left..(\alpha k)^{-\alpha}\left[1+\frac{-\alpha(2\beta+\alpha)}{2\alpha k}\right]+O\left(\frac{1}{(\alpha k)^3}\right)\right]|z|$$

$$\approx \frac{m^{mp}}{n^{nq}\,\alpha^{\alpha}}\frac{k^{pm}}{k^{qn+\alpha}}$$

then $\left|\frac{c_{k+1}}{c_k}\right| \to 0$ as $k \to \infty$ and $pm < qn + \mathrm{Re}(\alpha)$

which means that the $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)$ converges for all z provided that $pm < qn + \mathrm{Re}(\alpha)$

if $pm = qn + \mathrm{Re}(\alpha)$, then the series converges for $|z| < \delta = \alpha^{\alpha}$ and at the case of $|z| = \delta = \alpha^{\alpha}$ the series can converge on conditions depending on the parameter (see e.g. in [6])

Theorem 2.2:
If the condition in (1.4) is satisfied, then

$$\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z) = \beta\underset{p,q;m,n}{\overset{\alpha,\beta+1}{M}}(z) + \alpha z\frac{d}{dz}\underset{p,q;m,n}{\overset{\alpha,\beta+1}{M}} \tag{2.6}$$

Proof:

$$\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z) = \sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{z^k}{\Gamma(\alpha k+\beta)}$$

$$= \sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(a_q)_{kn}}\frac{(\alpha k+\beta)}{(\alpha k+\beta)}\frac{z^k}{\Gamma(\alpha k+\beta)}$$

$$= \sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(a_q)_{kn}}\frac{\alpha k\ z^k}{\Gamma(\alpha k+\beta+1)} + \sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{\beta z^k}{\Gamma(\alpha k+\beta+1)}$$

$$= \alpha z\sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(a_q)_{kn}}\frac{k\ z^{k-1}}{\Gamma(\alpha k+\beta+1)} + \beta\sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{z^k}{\Gamma(\alpha k+\beta+1)}$$

$$= \beta\underset{p,q;m,n}{\overset{\alpha,\beta+1}{M}}(z) + \alpha z\frac{d}{dz}\underset{p,q;m,n}{\overset{\alpha,\beta+1}{M}}(z) \qquad \text{which is (2.6)}$$

Theorem 2.3:
If the condition (1.4) is satisfied, then for $s \in \square$

$$\left(\frac{d}{dz}\right)^s\left[z^{\beta-1}\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(wz^{\alpha})\right] = z^{\beta-s-1}\underset{p,q;m,n}{\overset{\alpha,\beta-s}{M}}(wz^{\alpha}) \tag{2.7}$$

Proof:
Beginning with

$$\left(\frac{d}{dz}\right)^s\left[z^{\beta-1}\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(wz^{\alpha})\right] = \sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{w^k}{\Gamma(\alpha k+\beta)}\left(\frac{d}{dz}\right)^s\left(z^{\alpha k+\beta-1}\right)$$

$$=\sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{w^k}{\Gamma(\alpha k+\beta)}(\alpha k+\beta-1)(\alpha k+\beta-2)\ldots(\alpha k+\beta-s)z^{\alpha k+\beta-s-1}$$

$$=z^{\beta-s-1}\sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{w^k}{\Gamma(\alpha k+\beta-s)}z^{\alpha k}=z^{\beta-s-1}\underset{p,q;m,n}{\overset{\alpha,\beta-s}{M}}\left(wz^{\alpha}\right)$$

Theorem 2.4:
If the condition (1.4) is satisfied, then

$$\int_0^z t^{\beta-1}\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}\left(wt^{\alpha}\right)=z^{\beta}\underset{p,q;m,n}{\overset{\alpha,\beta+1}{M}}\left(wz^{\alpha}\right) \tag{2.8}$$

Proof:

$$\int_0^z t^{\beta-1}\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}\left(wt^{\alpha}\right)dt=\sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{w^k}{\Gamma(\alpha k+\beta)}\int_0^z t^{\alpha k+\beta-1}dt$$

$$z^{\beta}\sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{w^k}{\Gamma(\alpha k+\beta+1)}=z^{\beta}\underset{p,q;m,n}{\overset{\alpha,\beta+1}{M}}\left(wz^{\alpha}\right)$$

3. $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)$ IN TERMS OF OTHER FUNCTIONS

In this section we write $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)$ in terms of Wright generalized function, generalized hypergeometric function, Mellin Barnes integral and Fox's H-function.

$$\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)=\sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{z^k}{\Gamma(\alpha k+\beta)}$$

$$=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+km)}{\Gamma(a_1)}\cdots\frac{\Gamma(a_p+km)}{\Gamma(a_p)}\frac{\Gamma(b_1)}{\Gamma(b_1+kn)}\cdots\frac{\Gamma(b_q)}{\Gamma(b_q+kn)}\cdot\frac{\Gamma(k+1)}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}$$

$$=\frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\sum_{k=0}^{\infty}\frac{\Gamma(a_1+km)\cdots\Gamma(a_p+km)}{\Gamma(b_1+kn)\cdots\Gamma(b_q+kn)}\frac{\Gamma(k+1)}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}$$

Hence, we can write $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)$ in terms of the Wright generalized functions

$$\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)=\frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}\,{}_{p+1}\Psi_{q+1}\left[\begin{matrix}(a_1,m),\ldots,(a_p,m),(1,1)\\(b_1,n),\ldots,(b_q,n),(\beta,\alpha)\end{matrix};\ z\right] \tag{3.1}$$

Theorem 3.1:

Let (1.4) be satisfied with $\alpha=s\in\mathbb{N}$, then $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)$ can be written in terms of the generalized hypergeometric functions as

$$\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)=\frac{1}{\Gamma(\beta)}\,{}_{mp+1}F_{nq+s}\left[\begin{matrix}1,\Delta(m,a_1),\ldots,\Delta(m,a_p)\\\Delta(s,\beta),\Delta(n,b_1),\ldots,\Delta(n,b_q)\end{matrix};\frac{m^{mp}}{n^{nq}}\frac{z}{\alpha^{\alpha}}\right] \tag{3.2}$$

where $\Delta(s,\beta)$ is S-tuple $\frac{\beta}{s},\frac{\beta+1}{s},\ldots,\frac{\beta+s-1}{s}$

Proof:

Let $\alpha = s \in$ □ , then

$$\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z) = \frac{1}{\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{(a_1)_{km} \cdots (a_p)_{km}}{(b_1)_{kn} \cdots (b_q)_{kn}} \frac{z^k}{(\beta)_{\alpha k}}$$

$$= \frac{1}{\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{m^{mk} \prod_{r=1}^{m} \left(\frac{a_1 + r - 1}{m}\right)_k \cdots m^{mk} \prod_{r=1}^{m} \left(\frac{a_p + r - 1}{m}\right)_k \quad (1)_n \quad z^n}{n^{nk} \prod_{j=1}^{n} \left(\frac{b_1 + j - 1}{n}\right)_k \cdots n^{nk} \prod_{j=1}^{n} \left(\frac{b_q + j - 1}{n}\right)_k \quad . \quad \alpha^{\alpha k} \prod_{i=1}^{\alpha} \left(\frac{\beta + i - 1}{\alpha}\right)_k \quad n!}$$

$$= \frac{1}{\Gamma(\beta)} {}_{mp+1}F_{nq+s} \left[\begin{matrix} 1, \Delta(m, a_1), \ldots, \Delta(m, a_p) \\ \Delta(s, \beta), \Delta(n, b_1), \ldots, \Delta(n, b_q) \end{matrix} ; \frac{m^{mp}}{n^{nq}} \frac{z}{\alpha^{\alpha}} \right]$$

Now in order to write $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)$ in terms of Fox's H-function, we first express $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)$ as Mellin – Barnes type integral.

Theorem 3.2:

Let (1.4) be satisfied then $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)$ is represented in the MellinBranes type integral as

$$\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z) = \frac{1}{2\pi i} \int_L \frac{\Gamma(s)\Gamma(1-s)\Gamma(a_1 - ms) \ldots \Gamma(a_p - ms)(-z)^{-s} \, ds}{\Gamma(\beta - \alpha s)\Gamma(b_1 - ns) \ldots \Gamma(b_q - ns)} \tag{3.3}$$

Where $|ag(z)| < \pi$; the contour of integration begins at the and ending at $i\infty$ and intended to separate the poles of integrand at $s = -n$ for all $n \in$ □ (the left) from those at

$s = \left(\frac{a_i + k}{m}\right)_1^p$ and at $s = \left(\frac{b_j + k}{n}\right)_1^q$ for all $n \in$ □ $\bigcup \{0\}$ (to the right)

Proof:

evaluating

$$\frac{1}{2\pi i} \int_L \frac{\Gamma(s)\Gamma(1-s)\Gamma(a_1 - ms) \ldots \Gamma(a_p - ms)(-z)^{-s} \, ds}{\Gamma(\beta - \alpha s)\Gamma(b_1 - ns) \ldots \Gamma(b_q - ns)}$$

as the sum of residues at the poles $s = 0, -1, -2, \ldots$ we get

$$\sum_{k=0}^{k} \operatorname*{Res}_{s=-k} \left[\frac{\Gamma(s)\Gamma(1-s)\Gamma(a_1 - ms) \ldots \Gamma(a_p - ms)(-z)^{-s}}{\Gamma(\beta - \alpha s)\Gamma(b_1 - ns) \ldots \Gamma(b_q - ns)} \right]$$

$$= \sum_{k=0}^{k} \lim_{s+k \to 0} \frac{(-1)^k \pi(s+k)}{\sin \pi(s+k)} \left[\frac{\Gamma(a_1 - ms)\Gamma(a_2 - ms) \ldots \Gamma(a_p - ms)(-z)^{-s}}{\Gamma(\beta - \alpha s)\Gamma(b_1 - ns) \ldots \Gamma(b_q - ns)} \right]$$

$$= \frac{\Gamma(a_1) \ldots \Gamma(a_p)}{\Gamma(b_1) \ldots \Gamma(b_q)} \sum_{k=0}^{k} \frac{\Gamma(a_1)_{km} \cdots (a_p)_{km}}{(b_1)_{kn} \cdots (b_q)_{kn}} \cdot \frac{z^k}{\Gamma(\alpha k + \beta)}$$

hence

$$\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z) = \frac{\Gamma(b_1) \ldots \Gamma(b_q)}{\Gamma(a_1) \ldots \Gamma(a_p)} \cdot \frac{1}{2\pi i} \int_L \frac{\Gamma(s)\Gamma(1-s)\Gamma(a_1 - ms) \ldots \Gamma(a_p - ms)(-z)^{-s} \, ds}{\Gamma(\beta - \alpha s)\Gamma(b_1 - ns) \ldots \Gamma(b_q - ns)}$$

$$\frac{\Gamma(b_1) \ldots \Gamma(b_q)}{\Gamma(a_1) \ldots \Gamma(a_p)} H^{1,p+1}_{p+1,q+1} \left[-z \left| \begin{matrix} (0,1), (1-a_1, m), \ldots, (1-a_p, m) \\ (0,1), (1-\beta, \alpha), (1-b_1, n), \ldots, (1-b_q, n) \end{matrix} \right. \right] \tag{3.4}$$

The last equation is just a representation of $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)$ in terms of Fox's H-function

4. INTEGRAL TRANSFORMS OF $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)$

In this section, the image of $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)$ under Beta, Laplace and Mellin Barnes transforms will stated and proved in the following theorems.

Theorem 4.1: (Beta transform)

$$B\left\{\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}\left(xz^{\sigma}\right);\gamma,\delta\right\}=\frac{\Gamma(\delta)\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}$$

$$\cdot{}_{p+2}\Psi_{q+2}\left[\begin{matrix}(1,1),(\gamma,\sigma),(a_1,m),...,(a_p,m)\\(\beta,\alpha),(\gamma+\delta,\sigma),(b_1,n),...,(b_q,n)\end{matrix};\ x\right] \tag{4.1}$$

when (1.4) is satisfied and $\operatorname{Re}(\delta)>0, \operatorname{Re}(\gamma)>0$

Proof:

$$B\left\{\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}\left(xz^{\sigma};\gamma,\delta\right)\right\}=\int_0^1 z^{\gamma-1}(1-z)^{\delta-1}\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}\left(xz^{\sigma}\right)dz$$

$$=\sum_{k=0}^{\infty}\frac{\Gamma(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{x^k}{\Gamma(\alpha k+\beta)}\int_0^1 z^{\sigma k+\gamma-1}(1-z)^{\delta-1}dz$$

$$=\sum_{k=0}^{\infty}\frac{\Gamma(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{x^k}{\Gamma(\alpha k+\beta)}\ \beta(\sigma k+\gamma,\delta)$$

$$=\frac{\Gamma(\delta)\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}\sum_{k=0}^{\infty}\frac{\Gamma(a_1+km)\ldots\Gamma(a_p+km)\Gamma(\gamma+\sigma k)\Gamma(1+k)}{\Gamma(b_1+kn)\ldots\Gamma(b_q+kn)\Gamma(\alpha k+\beta)\Gamma(\gamma+\delta,\sigma)}\cdot\frac{x^k}{k!}$$

$$=\frac{\Gamma(\delta)\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}\ {}_{p+2}\Psi_{q+2}\left[\begin{matrix}(1,1),(\gamma,\sigma),(a_1,m),\ldots,(a_p,m)\\(\beta,\alpha),(\gamma+\delta,\sigma),(b_1,n),\ldots,(b_q,n)\end{matrix};x\right]$$

Theorem 4.2: (Laplace transform)

$$\mathcal{L}\left\{z^{\gamma-1}\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}\left(xz^{\sigma}\right);s\right\}=\frac{s^{-\gamma}\Gamma(b_1),\ldots,\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}$$

$${}_{p+2}\Psi_{q+1}\left[\begin{matrix}(a_1,m),\ldots,(a_p,m),(\gamma,\sigma),(1,1)\\(b_1,n),\ldots,(b_q,n),(\beta,\alpha)\end{matrix};\ \frac{x}{s^{\sigma}}\right] \tag{4.2}$$

Proof:

$$\mathcal{L}\left\{z^{\gamma-1}\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}\left(xz^{\sigma}\right);s\right\}=\int_0^{\infty} z^{\gamma-1}e^{-sz}\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}\left(xz^{\sigma}\right)dz$$

$$=\sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{x^k}{\Gamma(\alpha k+\beta)}\Gamma(\gamma+\sigma k)\int_0^{\infty}\frac{e^{-sz}z^{\gamma+\sigma k-1}}{\Gamma(\gamma+\sigma k)}dz$$

$$=\sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{x^k}{\Gamma(\alpha k+\beta)}\Gamma(\gamma+\sigma k)\ \mathcal{L}\left\{\frac{z^{\gamma+\sigma k-1}}{\Gamma(\gamma+\sigma k)};s\right\}$$

$$=\frac{s^{-\gamma}\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}\sum_{k=0}^{\infty}\frac{\Gamma(a_1+k_m)\ldots\Gamma(a_p+k_m)\Gamma(\gamma+\sigma_k)\Gamma(1+n)(xs^{-\sigma})^k}{\Gamma(b_1+k_n)\ldots\Gamma(b_q+k_n)\Gamma(\alpha_k+\beta)\qquad k!}$$

$$=\frac{s^{-\gamma}\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}\,{}_{p+2}\Psi_{q+1}\left[\begin{matrix}(a_1,m),\ldots,(a_p,m),(\gamma,\sigma),(1,1)\\(b_1,n),\ldots,(b_q,n),(\beta,\alpha)\end{matrix};\frac{x}{s^{\sigma}}\right]$$

Theorem 4.3: (Mellin transform)

$$\mathcal{M}\left[\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(-wz);s\right]=\frac{\Gamma(b_1)\ldots\Gamma(b_q)\Gamma(s)\Gamma(1-s)\Gamma(a_1-ms)\ldots\Gamma(a_p-ms)w^{-s}}{\Gamma(a_1)\ldots\Gamma(a_p)\Gamma(\beta-\alpha s)\Gamma(b_1-ns)\ldots\Gamma(b_q-ns)}\tag{4.3}$$

Proof:

According to theorem 3.2 and using (13), $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(-wz)$ can be written as

$$\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(-wz)=\frac{1}{2\pi i}\frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}\int_L\frac{\Gamma(s)\Gamma(1-s)\Gamma(a_1-ms)\ldots\Gamma(a_p-ms)(wz)^{-s}}{\Gamma(\beta-\alpha s)\Gamma(b_1-ns)\ldots\Gamma(b_q-ns)}ds$$

where $f^*(s)=\frac{\Gamma(s)\Gamma(1-s)\Gamma(a_1-ms)\ldots\Gamma(a_p-ms)}{\Gamma(\beta-\alpha s)\Gamma(b_1-ns)\ldots\Gamma(b_q-ns)w^s}$ and L is the contour of integration that being at $c-i\infty$ and ends at $c+i\infty;c\in\mathbb{R}$

hence $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(-wz)=\frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}\mathcal{M}^{-1}\{f^*(s);z\}$

now applying Mellin transform to both sides , we get

$$\mathcal{M}\left[\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(-wz);s\right]=\frac{\Gamma(b_1)\ldots(b_q)\Gamma(s)\Gamma(1-s)\Gamma(a_1-ms)\ldots\Gamma(a_p-ms)w^{-s}}{\Gamma(a_1)\ldots\Gamma(a_p)\Gamma(\beta-\alpha s)\Gamma(b_1-ns)\ldots\Gamma(b_q-ns)}$$

which proves (4.3)

5. FRACTIONAL CALCULUS GENERATING M-SERIES

In this section we consider composition of the Riemann – Liouville fractional integral and derivative and Hilfer's fractional derivative (1.6) – (1.8) with $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(z)$ defined by (1.3)

Theorem 5.1:

Let $c\in\mathbb{R}_+$, $\alpha,\beta\in\mathbb{C}$ with $\mathrm{Re}(\alpha)>0$ and $m,n>0$ then for $x>c$ we have

$$I_{c^+}^{\lambda}\left[(t-c)^{\beta-1}\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}\left(w(t-c)^{\alpha}\right)\right](x)=(x-c)^{\beta+\lambda-1}\underset{p,q;m,n}{\overset{\alpha,\beta+\lambda}{M}}\left(w(t-c)^{\alpha}\right)\tag{5.1}$$

Proof:

Beginning with $I_{c^+}^{\lambda}\left[(t-a)^{\beta-1}\right](x)=\frac{\Gamma(\beta)}{\Gamma(\lambda+\beta)}(x-a)^{\beta+\lambda-1}$, then

$$I_{c^+}^{\lambda}\left[(t-c)^{\beta-1}\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}\left(w(t-c)^{\alpha}\right)\right](x)=\sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{w^k}{\Gamma(\alpha k+\beta)}I_{c^+}^{\lambda}[t-c]^{\alpha k+\beta-1}$$

$$=\sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{w^k}{\Gamma(\alpha k+\beta)}\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha k+\beta+\lambda)}[x-c]^{\alpha k+\beta+\lambda-1}$$

$$(x-c)^{\beta+\lambda-1}\underset{p,q;m,n}{\overset{\alpha,\beta+\lambda}{M}}\left(w(x-c)^{\alpha}\right)$$

Theorem 5.2:
If the condition of theorem 5.1 is satisfied, then

$$D_{c^+}^{\lambda}\left[(t-c)^{\beta-1}\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}}\left(w\,(t-c)^{\alpha}\right)\right](x)=(x-c)^{\beta-\lambda-1}\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta-\lambda}{M}}\left(w\,(x-c)^{\alpha}\right) \tag{5.2}$$

Proof:
Beginning with def. of $D_{c^+}^{\lambda}$ in (1.7)

$$D_{c^+}^{\lambda}\left[(t-c)^{\beta-1}\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}}\left(w\,(t-c)^{\alpha}\right)\right](x)=\left(\frac{d}{dx}\right)^{s}\left[I_{c^+}^{s-\lambda}(t-c)^{\beta-1}\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}}\left(w\,(t-c)^{\alpha}\right)\right](x)$$

Now making use of (5.1) and (2.7), yields

$$=\left(\frac{d}{dx}\right)^{s}\left[(x-c)^{\beta+s-\lambda-1}\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta+s-\lambda}{M}}\left(w\,(x-c)^{\alpha}\right)\right]$$

$$=(x-c)^{\beta-\lambda-1}\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta-\lambda}{M}}\left(w\,(x-c)^{\alpha}\right)$$

Now we can get a similar result concerning the composition of Hilffer's fractional derivative with (1.3) contained in

Theorem 5.3:
Let $c\in\mathbb{R}_+$, $\alpha,\beta\in\mathbb{C}$ with $\mathrm{Re}(\alpha)>0$ and $m,n>0$, $0<u<1$, $o\le v\le 1$ and $\mathrm{Re}(\beta)>u+v-uv$ then for $x>c$

$$D_{c^+}^{u,v}\left[(t-c)^{\beta-1}\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}}\left(w\,(t-c)^{\alpha}\right)\right](x)=(x-c)^{\beta-u-1}\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta-u}{M}}\left(w\,(x-c)^{\alpha}\right) \tag{5.3}$$

Proof:
Beginning with $D_{c^+}^{u,v}\left[(t-c)^{\beta-1}\right]=\dfrac{\Gamma(\beta)}{\Gamma(\beta-u)}(x-c)^{\beta-u-1}$, then

$$D_{c^+}^{u,v}\left[(t-c)^{\beta-1}\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}}\left(w\,(t-c)^{\alpha}\right)\right](x)=\sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{w^{k}}{\Gamma(\alpha k+\beta)}D_{c^+}^{u,v}\left[(t-c)^{\alpha k+\beta-1}\right](x)$$

$$=\sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{w^{k}}{\Gamma(\alpha k+\beta)}\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha k+\beta-u)}(x-c)^{\alpha k+\beta-u-1}$$

$$=(x-c)^{\beta-u-1}\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta-u}{M}}\left(w\,(x-c)^{\alpha}\right)$$

6. INTEGRAL OPERATOR WITH GENERALIZED M-SERIES IN THE KERNEL

Consider the integral operator defined in (1.5) containing $\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}}(z)$ in the kernel. First of all we will prove the operator $\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta}{\mathcal{M}}}$ is bounded on $L(a,b)$

Theorem 6.1:

Let $\alpha,\beta\in\mathbb{C}$ with $\mathrm{Re}(\alpha)>0$, $m,n>0$ and $b>a$, then the operator $\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta}{\mathcal{M}}}$ is bounded on $L(a,b)$ and

$$\left\|\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta}{\mathcal{M}}}\varphi\right\|_1\le\beta\|\varphi\|_1 \tag{6.1}$$

where,

$$\beta=(b-a)^{\mathrm{Re}(\beta)}=\sum_{k=0}^{\infty}\frac{|(a_1)_{km}|\cdots|(a_p)_{km}|}{|(b_1)_{kn}|\cdots|(b_q)_{kn}|}\frac{\left|w\,(b-a)^{\mathrm{Re}(\alpha)}\right|^{k}}{|\Gamma(\alpha k+\beta)|} \tag{6.2}$$

Proof:

First of all, let c_k denote the k^{th} term of (45), then

$$\left|\frac{c_{k+1}}{c_k}\right| = \left|\frac{(a_1)_{km+m}}{(a_1)_{km}}\right| \cdots \left|\frac{(a_p)_{km+m}}{(a_p)_{km}}\right| \left|\frac{(b_1)_{kn}}{(b_1)_{kn+n}}\right| \cdots \left|\frac{(b_q)_{kn}}{(b_q)_{kn+n}}\right| \left|\frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha k+\beta+\alpha)}\right|$$

$$\left|\frac{\mathrm{Re}(\alpha)k+\mathrm{Re}(\beta)}{\mathrm{Re}(\alpha)k+\mathrm{Re}(\alpha)+\mathrm{Re}(\beta)}\right| \left|w\,(b-a)^{\mathrm{Re}(\alpha)}\right|$$

$$\approx \frac{\left|w\,(b-a)^{\mathrm{Re}(\alpha)}\right| \quad (km)^{mp}}{(|\alpha|k)^{\alpha} \qquad (kn)^{nq}} \qquad \text{as } k \to \infty$$

Hence $\left|\frac{c_{k+1}}{c_k}\right| \to 0$ as $k \to \infty$ and $mp < nq + \mathrm{Re}(\alpha)$ which means that the right hand side of (6.2) is convergent and finite under the given condition.

Now according to (1.5) and (1.17)

$$\left\| \underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta}{\mathcal{M}}} \varphi \right\|_1 = \int_c^b \left[\int_c^x (x-t)^{\beta-1} \underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}} \left(w\,(x-t)^{\alpha}\right) \right] |\varphi(t)dt|dx$$

$$\le \int_c^b \left[\int_t^b (x-t)^{\beta-1} \left| \underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}} \left(w\,(x-t)^{\alpha}\right) \right| dx \right] |\varphi(t)|dt$$

and by putting $u = x - t$

$$= \int_c^b \left[\int_0^{b-t} u^{\mathrm{Re}(\beta)-1} \left| \underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}} (wu^{\alpha}) \right| du \right] |\varphi(t)|dt$$

$$\le \int_c^b \left[\int_0^{b-a} u^{\mathrm{Re}(\beta)-1} \left| \underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}} (wu^{\alpha}) \right| du \right] |\varphi(t)|dt$$

but we have

$$\int_0^{b-a} u^{\mathrm{Re}(\beta)-1} \left| \underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}} (wu^{\alpha}) \right| du = \sum_{k=0}^{\infty} \frac{|(a_1)_{km}| \ldots |(a_p)_{km}|}{|(b_1)_{kn}| \ldots |(b_q)_{kn}|} \frac{w^n}{|\Gamma(\alpha k+\beta)|} \int_0^{b-a} u^{\mathrm{Re}(\alpha)n+\mathrm{Re}(\beta)-1} du = \beta$$

so that

$$\beta = (b-a)^{\mathrm{Re}(\beta)} \sum_{k=0}^{\infty} \frac{|(a_1)_{km}| \ldots |(a_p)_{km}|}{|(b_1)_{kn}| \ldots |(b_q)_{kn}|} \frac{\left|w\,(b-a)^{\mathrm{Re}(\alpha)}\right|^k}{|\Gamma(\alpha k+\beta)|\,|\mathrm{Re}(\alpha)k+\mathrm{Re}(\beta)|}$$

Hence

$$\left\| \underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta}{\mathcal{M}}} \varphi \right\|_1 \le \int_a^b \beta |\varphi(t)|dt = \beta \|\varphi\|_1$$

Another result stated an proved in the next corollary.

Corollary 6.2:

$$\left[\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta}{\mathcal{M}}} (t-c)^{\xi-1} \right](x) = \Gamma(\xi)(x-c)^{\beta+\xi-1} \underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta+\xi}{M}} \left[w\,(x-c)^{\alpha}\right] \qquad (6.3)$$

Proof:

Making use of (1.5) and (1.9) yields.

$$\left[\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta}{\mathcal{M}}} (t-c)^{\xi-1} \right](x) = \int_c^x (x-t)^{\beta-1} \underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}} \left[w\,(x-t)^{\alpha}\right] (t-c)^{\xi-1} dt$$

$$= \sum_{k=0}^{\infty} \frac{(a_1)_{km} \cdots (a_p)_{km}}{(b_1)_{kn} \cdots (b_q)_{kn}} \frac{w^n}{\Gamma(\alpha k+\beta)} \int_c^x (x-t)^{\alpha\beta+\beta-1} (t-c)^{\xi-1} dt$$

Setting $t = c + y\left(x - c\right)$, we get

$$\left[\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta}{\mathcal{M}}}\left(t-c\right)^{\xi-1}\right]\left(x\right)=\sum_{k=0}^{\infty}\frac{\left(a_1\right)_{km}\cdots\left(a_p\right)_{km}}{\left(b_1\right)_{kn}\cdots\left(b_q\right)_{kn}}\frac{w^k\left(x-c\right)^{\alpha k+\beta+\xi-1}}{\Gamma\left(\alpha k+\beta\right)}\int_0^1 y^{\xi-1}\left(1-y\right)^{\alpha k+\beta-1}dy$$

$$=\sum_{k=0}^{\infty}\frac{\left(a_1\right)_{km}\cdots\left(a_p\right)_{km}}{\left(b_1\right)_{kn}\cdots\left(b_q\right)_{kn}}\frac{w^k\left(x-c\right)^{\alpha k+\beta+\xi-1}}{\Gamma\left(\alpha k+\beta\right)}\beta\left(\alpha k+\beta,\xi\right)$$

$$=\Gamma\left(\xi\right)\left(x-c\right)^{\beta+\xi-1}\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta+\xi}{M}}\left[w\left(x-c\right)^{\alpha}\right]$$

7. COMPOSITION OF FRACTIONAL CALCULUS OPERATORS AND INTEGRAL OPERATORS WITH GENERALIZED M-SERIS IN THE KERNEL

We consider now the composition of Riemann – Liouville fractional integration and differentiation operators $I_{c^+}^{\lambda}$, $D_{c^+}^{\lambda}$ and Hilfer's fractional derivative $D_{c^+}^{u,v}$ with the operator $\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta}{\mathcal{M}}}$ defined in (1.5)

Theorem 7.1:
Let $\alpha,\beta\in\mathbb{C}$ with $\mathrm{Re}\left(\alpha\right)>0$ and $m,n>0$ then

$$I_{c^+}^{\lambda}\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta}{\mathcal{M}}}\varphi=\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta+\lambda}{\mathcal{M}}}\varphi=\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta}{\mathcal{M}}}I_{c^+}^{\lambda}\varphi \tag{7.1}$$

holds for any summable function $\varphi\in L\left(a,b\right)$

Proof:

Making use of (1.5), (1.6) and (1.17), we get

$$\left(I_{c^+}^{\lambda}\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta}{\mathcal{M}}}\varphi\right)\left(x\right)=\frac{1}{\Gamma\left(\lambda\right)}\int_c^x\left(x-u\right)^{\lambda-1}\left[\int_c^u\left(u-t\right)^{\beta-1}\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}}\left[w\left(u-t\right)^{\alpha}\right]\varphi\left(t\right)dt\right]du$$

$$=\int_c^x\left[\frac{1}{\Gamma\left(\lambda\right)}\int_t^x\left(x-u\right)^{\lambda-1}\left(u-t\right)^{\beta-1}\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}}\left[w\left(u-t\right)^{\alpha}\right]du\right]\varphi\left(t\right)dt$$

letting $\tau=u-t$

$$\left(I_{c^+}^{\lambda}\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta}{\mathcal{M}}}\varphi\right)\left(x\right)=\int_c^x\left[\frac{1}{\Gamma\left(\lambda\right)}\int_0^{x-t}\left(x-t-\tau\right)^{\lambda-1}\tau^{\beta-1}\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}}\left(w\,\tau^{\alpha}\right)d\tau\right]\varphi\left(t\right)dt$$

$$=\int_c^x I_0^{\lambda}\left[\tau^{\beta-1}\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}}\left(w\,\tau^{\alpha}\right)\right]\left(x-t\right)\varphi\left(t\right)dt=\int_c^x\left[\tau^{\beta+\lambda-1}\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta}{M}}\left(w\,\tau^{\alpha}\right)\right]\varphi\left(t\right)\left(t\right)dt$$

$$=\int_c^x\left[\left(x-t\right)^{\beta+\lambda-1}\underset{p,q;\mathrm{m,n}}{\overset{\alpha,\beta+\lambda}{M}}\left(w\left(x-t\right)^{\alpha}\right)\right]\varphi\left(t\right)dt=\left(\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta+\lambda}{\mathcal{M}}}\varphi\right)\left(x\right)$$

Similarly, we can prove the other side·

Theorem 7.2:
If the condition of theorem 7.1 be satisfied, then

$$\left(D_{c^+}^{\lambda}\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta}{\mathcal{M}}}\varphi\right)\left(x\right)=\left(\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta-\lambda}{\mathcal{M}}}\varphi\right)\left(x\right) \tag{7.2}$$

Proof:
Let $s=\left[\mathrm{Re}\,\lambda\right]+1$ and using (1.7) and (5.1), we get

$$\left(D_{c^+}^{\lambda}\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta}{\mathcal{M}}}\varphi\right)\left(x\right)=\left(\frac{d}{dx}\right)^{s}\left(I_{c^+}^{s-\lambda}\underset{p,q;\mathrm{m,n},w,c^+}{\overset{\alpha,\beta}{\mathcal{M}}}\varphi\right)\left(x\right)$$

$$=\left(\frac{d}{dx}\right)^{s}\left(\underset{p,q;m,n,w,c^{+}}{\overset{\alpha,\beta+s-\lambda}{\mathcal{M}}}\varphi\right)(x)$$

$$=\left(\frac{d}{dx}\right)^{n}\int_{a}^{x}(x-t)^{\beta+s-\lambda-1}\underset{p,q;m,n,w,c^{+}}{\overset{\alpha,\beta+s-\lambda}{\mathcal{M}}}\left[w(x-t)^{\alpha}\right]\varphi(t)dt$$

Since the integral is continuous, using (1.18) yields.

$$\left(D_{c^{+}}^{\lambda}\underset{p,q;m,n,w,c^{+}}{\overset{\alpha,\beta}{\mathcal{M}}}\varphi\right)(x)=\left(\frac{d}{dx}\right)^{s-1}\int_{c}^{x}\frac{\partial}{\partial x}\left[(x-t)^{\beta+s-\lambda-1}\underset{p,q;m,n}{\overset{\alpha,\beta+s-\lambda}{M}}\left[w(x-t)^{\alpha}\right]\right]\varphi(t)dt$$

$$=\left(\frac{d}{dx}\right)^{n-1}\int_{c}^{x}(x-t)^{\beta+s-\lambda-2}\left(\sum_{k=0}^{\infty}\frac{(a_1)_{km}\cdots(a_p)_{km}}{(b_1)_{kn}\cdots(b_q)_{kn}}\frac{\left[w(x-t)^{\alpha}\right]^{k}}{\Gamma(\alpha k+\beta+s-\lambda-1)}\right)\varphi(t)dt$$

$$=\left(\frac{d}{dx}\right)^{n-1}\int_{c}^{x}(x-t)^{\beta+s-\lambda-2}\underset{p,q;m,n}{\overset{\alpha,\beta+n-\lambda-1}{M}}\left[w(x-t)^{\alpha}\right]\varphi(t)dt$$

Repeating this process $(k-1)$ times, then we get

$$\left(D_{c^{+}}^{\lambda}\underset{p,q;m,n,w,c^{+}}{\overset{\alpha,\beta}{\mathcal{M}}}\varphi\right)(x)=\int_{c}^{x}\left[(x-t)^{\beta-\lambda-1}\underset{p,q;m,n}{\overset{\alpha,\beta-\lambda}{M}}\left[w(x-t)^{\alpha}\right]\right]\varphi(t)dt=\left(\underset{p,q;m,n,w,c^{+}}{\overset{\alpha,\beta-\lambda}{\mathcal{M}}}\varphi\right)(x)$$

Theorem 7.3:

Let $\alpha,\beta\in\mathbb{C}$ with $\mathrm{Re}(\alpha)>0$, $0<u<1$, $o\le v\le 1$, $\mathrm{Re}(\beta)>u+v-uv$ and $m,n>0$, then

$$\left(D_{c^{+}}^{u,v}\underset{p,q;m,n,w,c^{+}}{\overset{\alpha,\beta}{\mathcal{M}}}\varphi\right)(x)=\left(\underset{p,q;m,n,w,c^{+}}{\overset{\alpha,\beta-u}{\mathcal{M}}}\varphi\right)(x)\qquad(7.3)$$

Proof:

Making use of (7.2) instead of λ put $u+v-uv$, we get

$$\left(D_{c^{+}}^{u+v-uv}\underset{p,q;m,n,w,c^{+}}{\overset{\alpha,\beta}{\mathcal{M}}}\varphi\right)(x)=\left(\underset{p,q;m,n,w,c^{+}}{\overset{\alpha,\beta-u-v+uv}{\mathcal{M}}}\varphi\right)(x)$$

also from (1.8) we have

$$\left(D_{c^{+}}^{u,v}\varphi\right)(x)=\left(I_{c^{+}}^{v(1-u)}\left(\frac{d}{dx}\right)\left[I_{c^{+}}^{(1-v)(1-u)}\varphi\right]\right)(x)=\left(I_{c^{+}}^{v(1-u)}\left(\frac{d}{dx}\right)\left[I_{c^{+}}^{1-v-u+uv}\varphi\right]\right)(x)$$

Now making use (7.2) again, yields.

$$\left[D_{c^{+}}^{u,v}\underset{p,q;m,n,w,c^{+}}{\overset{\alpha,\beta}{\mathcal{M}}}\varphi\right](x)=\left(I_{c^{+}}^{v(1-u)}\left[D_{c^{+}}^{u+v-uv}\underset{p,q;m,n,w,c^{+}}{\overset{\alpha,\beta}{\mathcal{M}}}\varphi\right]\right)(x)=\left(I_{c^{+}}^{v(1-u)}\underset{p,q;m,n,w,c^{+}}{\overset{\alpha,\beta-u-v+uv}{\mathcal{M}}}\varphi\right)(x)$$

$$=\left(\underset{p,q;m,n,w,c^{+}}{\overset{\alpha,\beta-u-v+uv+v(1-u)}{\mathcal{M}}}\varphi\right)(x)=\left(\underset{p,q;m,n,w,c^{+}}{\overset{\alpha,\beta-u}{\mathcal{M}}}\varphi\right)(x)$$

8. REFERENCES

1. Fox, H., The G – and H-function as symmetric Fourier kernels. Trans. Amer. Math. Soc. 98, 395-429 (1961).
2. Hilfer, R ., Application of fractional calculus in physics. Singapore, New Jersy; London and HongKong. World scientific Publishing Company, (2000).
3. Inayat – Hussain, A., New properties of hypergeometric series derivable from Feynman integrals, II: Ageneralization of the H-function. J. Phys. A: Math. Gen. 20, 4119-4128 (1987).
4. Kilbas, A , and Saigo, M., H-transforms: Theory and applications, London, New York: Chapman and Hall/ CRS, (2004).
5. Kilbas, A , Saigo , M , and Sexena, R., Generalized Mittag-Leffler function and generalized fractional calculus operators. Integral Transforms Spec. Funct., Vol.15, 31-49 (2004).

6. Kilbas, A , Srivastava, H , and Trujillo, J., Theory and applications of fractional differential equations. Elsevier, North Holland Math. Studies 204, Amsterdam; etc, (2006).
7. Kiryakova, V., The special functions of fractional calculus as generalized fractional calculus operators of some basic functions. Comp. Math. Appl. 59(3),1128-1141(2010).
8. Mathai, A , and Sexena, R ., The H-function with applications in statistics and other disciplines. John Wiley and Sons, New York (1978).
9. Prudnikov, A , Bbrychkov, Yu, and Marichev , O ., Integrals and Series. Vol.3: More special functions. Gordon and Breach, New York NJ (1990).
10. Rainville, E ., Special functions. New York: Chelsea Pulb. Co; (1960).
11. Salim, T, and Faraj, A ., A generalization of Mittag – Leffler function and integral operator associated with fractional calculus. J. Frac. Cal. appl. Vol.3, No.5, 1-13 (2012).
12. Samko, S, Kilbas, A and Marichev, O., Fractional Integrals and derivatives. Theory and applications. Gordon and Breach, Sci. Publ New York (1993).
13. Sexena, R ., A remark on a paper on M-series. Fract. Calc. Appl. Anal.12, No.1, 109-110 (2009).
14. Sharma, M., Fractional integration and fractional differentiation of the M-series. Fract. Calc. appl. Anal. 11, No.2, 187-192 (2008).
15. Sharma, M and Jain, R., A note on a generalized M-series as a special function of fractional calculus. Fract. Calc. appl. Anal. 12, No.4, 449-452 (2009).
16. Shukla, A , and Prajapati, J ., On a generalization of Mittag – Leffler function and its properties. J. Math. Appl. Anal. 336, 797-811 (2007).
17. Sneddon, I ., The use of integral transforms. Tata McGraw Hill .New Delhi , (1979).
18. Srivastava, H , and Manocha , H ., A treatise on generating functions. John Willy and Sons,New York , (1984).
19. Srivistave, H , and Tomovski, Z., Fractional calculus with an integral operator containing a generalized Mittag – Leffler function in the Kernel . Appl. Math. Comput., 211, 198-210 (2009).