Exponential growth of solution of a strongly nonlinear reaction diffusion equation *

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Abstract An initial boundary value problem for strongly nonlinear reaction diffusion equation is studied. We show the exponential growth of solution with L_p- norm using a differential inequalities.

Keywords nonlinear reaction diffusion equation, exponential growth, multiple nonlinearities

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1 Introduction

In this paper, we study the following the initial boundary value problem of a class of reaction diffusion equation with multiple nonlinearities

$$u_t - \Delta u + |u|^{k-2}u_t = |u|^{p-2}u, \tag{1.1}$$

$$u(x,t) = 0, x \in \partial\Omega, \tag{1.2}$$

$$u(x,0) = u_0(x), x \in \Omega, \tag{1.3}$$

where $k > 2, p > 2$ are real numbers and Ω is bounded domain in R^n with smooth boundary $\partial\Omega$ so that the divergence theorem can be applied. Here,Δ denotes the Laplace operator in Ω.

This type of problems are not only important from the theoretical point of view, but also arise in many physical applications and describe a great deal of models in applied science. It appears in the models of chemical reactions, heat transfer, population dynamics, and so on (see [1] and references therein).

In the absence of the nonlinear diffusion term $|u|^{k-2}u_t$, the equation (1.1) reduced to the following equation

$$u_t - \Delta u = |u|^{p-2}u, \tag{1.4}$$

A related problems to the equation (1.4) have attracted a great deal of attention in the last two decades, and many results have been appeared on the existence, blowup and asymptotic

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behavior of solutions. It is well known that the nonlinear $|u|^{p-2}u$ reaction term drives the solution of (1.4) to blow up in finite time and the diffusion term is known to yield existence of global solution if the reaction term is removed from the equation [2]. The more general equation

$$u_t - div(|\bigtriangledown u|^{m-2} \bigtriangledown u) = f(u), \tag{1.5}$$

has also attracted a great deal of people. The obtained results show that global existence and nonexistence depend roughly on m, the degree of nonlinearity in f, the dimension n, and the size of the initial data. See in this regard, the works of Levine[3], Kalantarov and Ladyzhenskaya[4], Levine et al.[5], Messaoudi[6], Liu et al.[7] and references therein. Pucci and Serrin [8] have been discussed the stability of the following equation

$$|u_t|^{l-2}u_t - div(|\bigtriangledown u|^{m-2} \bigtriangledown u) = f(u), \tag{1.6}$$

Levine et al. [5] got the global existence and nonexistence of solution for (1.6). Pang et. al[9, 10] and Berrimi [11] given the sufficient condition of blow-up result for certain solutions of (1.6) with positive or negative initial energy.

The class of equation (1.1) can also be as a special case of doubly nonlinear parabolic-type equations (or the porous medium equation)[12, 5]

$$\beta(u)_t - \Delta u = |u|^{p-2}u \tag{1.7}$$

if we take $\beta(u) = u + |u|^{m-2}u$. Such equation play an important role in physics and biology. It should be noted that the questions of the solvability,local and global in time, asymptotic behavior and blowup of initial boundary value problems and initial value problems for equation of the type (1.7) were investigated by many authors. We only mention the work [12, 13] for this class equation.

We should also point out that Polat[14] established a blow up result for the solution with vanishing initial energy of the following initial boundary value problem

$$u_t - u_{xx} + |u|^{m-2}u_t = |u|^{p-2}u.$$

They also given detail results of the the necessary and sufficient blow up conditions together with blow up rate estimates for the positive solution of the problem

$$(u^m)_t - \Delta u = f(u),$$

subject to various boundary conditions. Korpusov [15, 16] have been obtained sufficient conditions for the blowup for a finite time and condition of solvability for the following generalized Boussinesq equation

$$u_t - \Delta u - \Delta u_t + |u|^{m-2}u_t = u(u-\alpha)(u-\beta), \tag{1.8}$$

with initial boundary value (1.2) and (1.3) in R^3 for $\alpha, \beta > 0$ by the convex method [3, 4].

In this paper, we will investigate the (1.1)-(1.3). To the best of our knowledge, the problem of (1.1)-(1.3) have not been well studied. We will prove that the solutions in bounded and

$$||u_t||^2 + || \bigtriangledown u||^2 \to \infty, t \to +\infty.$$

In fact it will be proved that the L_p-norm of the solution grows as an exponential function. An essential tool of the proof is an idea used in [17, 18], which based on an auxiliary function (which is a small perturbation of the total energy), using a differential inequalities and obtaining the result. This article is organized as follows. Section 2 is concerned with some notations and statement of assumptions. In Section 3, we give and prove the result if the initial energy $E(0)$ of our solutions is negative (this means that our initial data are large enough). In Section 4, we give and prove the result if the initial energy $E(0) > 0$.

2 Preliminaries

In this section, we will give some notations and statement of assumptions for k, p, g. We denote $L^p(\Omega)$ by L^p, $H_0^1(\Omega)$ by H_0^1, the usual Soblev space. The norm and inner of $L^p(\Omega)$ are denoted by $||.||_p = ||.||_{L^p(\Omega)}$ and $(u, v) = \int_\Omega u(x)v(x)dx$, respectively. Especially, $||.|| = ||.||_{L^2(\Omega)}$ for $p = 2$.

For the number k and p, we assume that

$$2 < k < p \leq \frac{2(n-1)}{n-2}, if\ n \geq 3; 2 < k < p < +\infty, if\ n = 1, 2.$$

Similar to [14], we call $u(x, t)$ a weak solution of problem (1.1)-(1.3) on $\Omega \times [0, T)$, if

$$u \in C(0, T; H_0^1) \cap C^1(0, T; L^2),\ |u|^{k-2}u_t \in L^2(\Omega \times [0, T))$$

satisfying $u(x, 0) = u_0(x)$ and

$$\int_0^t \int_\Omega [\bigtriangledown u(s) \bigtriangledown v(s) + u_t(s)v(s) + |u|^{k-2}u_t v - |u|^{p-2}uv]dxds = 0, \quad \forall v \in C(0, T; H_0^1),\ \forall t \in [0, T).$$

In this paper, we always assume that the problem (1.1)-(1.3) exist a local solution.

Now, we introduce two functionals

$$E(t) = E(u) = \frac{1}{2}|| \bigtriangledown u||^2 - \frac{1}{p}||u||_p^p, \tag{2.1}$$

$$E(0) = \frac{1}{2}|| \bigtriangledown u_0||^2 - \frac{1}{p}||u_0||_p^p, \tag{2.2}$$

where $u \in H_0^1$. Multiplying Equation (1.1) by u_t and integrating over Ω, we have

$$E'(t) = -||u_t||^2 - \int_\Omega |u|^{k-2}u_t^2 dx. \tag{2.3}$$

3 Exponential growth of solution in case of $E(0) < 0$

In this section, we will prove the first main result. Our techniques of proof follow very carefully the techniques used in [17, 18].

Theorem 3.1 Suppose that the assumption about k, p hold, $u_0 \in H_0^1$ and u is a local solution of the system (1.1)-(1.3), $E(0) < 0$. Then the solution of the system (1.1)-(1.3) grows exponentially.

Proof We set

$$H(t) = -E(t). \tag{3.1}$$

By the definition of $H(t)$ and (2.3)

$$H'(t) = -E'(t) \geq 0. \tag{3.2}$$

Consequently, $E(0) < 0$, we have

$$H(0) = -E(0) > 0. \tag{3.3}$$

It is clear that by (3.2) and (3.3)

$$0 < H(0) \leq H(t). \tag{3.4}$$

By (3.1) and the expression of $E(t)$,

$$H(t) - \frac{1}{p}||u||_p^p = -\frac{1}{2}|| \bigtriangledown u||^2 < 0, \tag{3.5}$$

One implies

$$0 < H(0) \leq H(t) \leq \frac{1}{p}||u||_p^p. \tag{3.6}$$

Let us define the functional

$$L(t) = H(t) + \frac{\epsilon}{2}||u||^2. \tag{3.7}$$

By taking the time derivative of (3.7) and by (1.1),we have

$$\begin{aligned} L'(t) &= H'(t) + \epsilon \int_\Omega u u_t dx \\ &= ||u_t||^2 + \int_\Omega |u|^{k-2} u_t^2 dx + \epsilon ||u||_p^p - \epsilon || \bigtriangledown u||^2 - \epsilon \int_\Omega |u|^{k-2} u u_t dx. \end{aligned} \tag{3.8}$$

To estimate the last term in the right-hand side of (3.8), we use the following Young's inequality

$$ab \leq \delta^{-1} a^2 + \delta b^2,$$

so we have

$$\int_\Omega |u|^{k-2}uu_t dx = \int_\Omega |u|^{\frac{k-2}{2}}u_t|u|^{\frac{k-2}{2}}udx \leq \delta^{-1}\int_\Omega |u|^{k-2}u_t^2dx + \delta\int_\Omega |u|^k dx.$$

Therefore, we have

$$L'(t) \geq ||u_t||^2 - \epsilon||\bigtriangledown u||^2 + \epsilon||u||_p^p - \epsilon\delta||u||_k^k + (1-\epsilon\delta^{-1})\int_\Omega |u|^{k-2}u_t^2dx. \tag{3.9}$$

By using

$$||u||_p^p = pH(t) + \frac{p}{2}||\bigtriangledown u||^2,$$

Hence, (3.9) becomes

$$\begin{aligned} L'(t) &\geq ||u_t||^2 - \epsilon||\bigtriangledown u||^2 + \epsilon[pH(t) + \frac{p}{2}||\bigtriangledown u||^2] - \epsilon\delta||u||_k^k + (1-\epsilon\delta^{-1})\int_\Omega |u|^{k-2}u_t^2dx \\ &\geq ||u_t||^2 + (1-\epsilon\delta^{-1})\int_\Omega |u|^{k-2}u_t^2dx + \epsilon a_1||\bigtriangledown u||^2 + \epsilon pH(t) - \epsilon\delta||u||_k^k, \end{aligned} \tag{3.10}$$

where $a_1 = \frac{p}{2} - 1 > 0$. Note that $p > k > 2$ and embedding theorem,

$$||u||_k^k \leq C||u||_p^k \leq C(||u||_p^p)^{\frac{k}{p}},$$

where $C > 0$ is a positive constant. Since $0 < \frac{k}{p} < 1$, now applying the inequality $x^l \leq (x+1) \leq (1+\frac{1}{z})(x+z)$, which holds for all $x \geq 0, 0 \leq l \leq 1, z > 0$, in particular, taking $x = ||u||_p^p, l = \frac{k}{p}, z = H(0)$, we obtain

$$(||u||_p^p)^{\frac{k}{p}} \leq (1 + \frac{1}{H(0)})(||u||_p^p + H(0)),$$

then from (3.6)

$$||u||_k^k \leq C||u||_p^k \leq C_1||u||_p^p, \tag{3.11}$$

so we have

$$L'(t) \geq ||u_t||^2 + (1-\epsilon\delta^{-1})\int_\Omega |u|^{k-2}u_t^2dx + \epsilon a_1||\bigtriangledown u||^2 + \epsilon pH(t) - \epsilon\delta C_1||u||_p^p, \tag{3.12}$$

Taking $0 < 2a_2 = a_1$, and by $2H(t) \geq -||\bigtriangledown u||^2 + \frac{2}{p}||u||_p^p$, we have

$$\begin{aligned} L'(t) &\geq ||u_t||^2 + (1-\epsilon\delta^{-1})\int_\Omega |u|^{k-2}u_t^2dx + \epsilon(a_1 - a_2)||\bigtriangledown u||^2 \\ &+ \epsilon pH(t) - \epsilon\delta C_1||u||_p^p + \epsilon a_2[||\bigtriangledown u||^2 - \frac{2}{p}||u||_p^p] \\ &= ||u_t||^2 + (1-\epsilon\delta^{-1})\int_\Omega |u|^{k-2}u_t^2dx + \epsilon a_2||\bigtriangledown u||^2 \\ &+ \epsilon(\frac{2}{p}a_2 - \delta C_1)||u||_p^p + \epsilon(p - 2a_2)H(t). \end{aligned} \tag{3.13}$$

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Taking δ small enough such that $\frac{2}{p}a_2 - \delta C_1 > 0$, and then taking ϵ small enough such that $1 - \epsilon\delta^{-1} > 0$, and noting that $p - 2a_2 = p - a_1 = \frac{p}{2} + 1 > 0$, then

$$L'(t) \geq C_2(H(t) + ||u_t||^2 + || \bigtriangledown u||^2 + ||u||_p^p). \tag{3.14}$$

On the other hand, by the definition of $L(t)$, we get that there exists a positive constant C such that

$$\begin{aligned} L(t) &= H(t) + \frac{\epsilon}{2}||u||^2 \\ &\leq C_3(H(t) + || \bigtriangledown u||^2) \\ &\leq C_3(H(t) + ||u_t||^2 + || \bigtriangledown u||^2 + ||u||_p^p). \end{aligned} \tag{3.15}$$

From (3.14) and (3.15), we obtain the differential inequality

$$L'(t) \geq rL(t), \tag{3.16}$$

Integration of (3.16) between 0 and t gives us

$$L(t) \geq L(0)exp(rt), \tag{3.17}$$

From (3.7) and ϵ small enough, we have

$$L(t) \leq H(t) \leq \frac{1}{p}||u||_p^p. \tag{3.18}$$

By (3.17) and (3.18), we have

$$||u||_p^p \geq Cexp(rt).$$

Therefore, we conclude that the solution in the L_p-norm growths exponentially.

4 Exponential growth of solution in case of $E(0) \geq 0$

In this section, we will prove that the energy will grow up as an exponential function as time goes to infinity, provided that the initial energy $E(0) > 0$.

The following Lemma will play an essential role in the proof of our main result, and it is similar to a Lemma used firstly by Vitillaro [19]. In order to give the result and for the sake of simplicity, we introduce the following We set

$$\lambda_1 = C_*^{-\frac{p}{p-2}}, E_1 = (\frac{1}{2} - \frac{1}{p})\lambda_1^2.$$

Lemma 4.1 Let u be a solution of (1.1)-(1.3). Suppose that the assumption of k, p hold. Assume further that $E(0) < E_1$ and $||u_0|| > \lambda_1$. Then there exists a constant $\lambda_2 > \lambda_1$ such that $||u|| > \lambda_2$.

Let us now state our main result.

Theorem 4.1 Suppose that the assumption about k, p hold, $u_0 \in H_0^1$ and u is a local solution of the system (1.1)-(1.3), $||u_0|| > \lambda_1$ and $E(0) < E_1$. Then the solution of the system (1.1)-(1.3) grows exponentially.

Proof We set

$$H(t) = E_2 - E(t), \tag{4.1}$$

where $E(0) < E_2 < E_1$. By the definition of $H(t)$ and (2.3)

$$H'(t) = -E'(t) \geq 0. \tag{4.2}$$

Consequently,

$$H(0) = E_2 - E(0) > 0. \tag{4.3}$$

It is clear that by (4.2) and (4.3)

$$0 < H(0) \leq H(t). \tag{4.4}$$

By (4.1), the expression of $E(t)$, and Lemma 4.1

$$\begin{aligned} H(t) &= E_2 - \frac{1}{2}||\bigtriangledown u||^2 + \frac{1}{p}||u||_p^p \\ &\leq E_1 - \frac{1}{2}\lambda_1^2 + \frac{1}{p}||u||_p^p = -\frac{1}{p}\lambda_1^2 + \frac{1}{p}||u||_p^p < \frac{1}{p}||u||_p^p, \end{aligned} \tag{4.5}$$

One implies

$$0 < H(0) \leq H(t) \leq \frac{1}{p}||u||_p^p. \tag{4.6}$$

Then we can prove the theorem by the similar proof of Theorem 3.1.

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