Synchronization and Anti-Synchronization of a New Hyperchaotic System Using Nonlinear Control

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ABSTRACT— ***Synchronization is characterized by the equality of state variables while evolving in time. Anti-synchronization is categorized by the disappearance of the sum of relevant variables. This paper has studied and investigated the global chaos identical Synchronization and Anti-Synchronization (AS) of a new hyperchaotic system via Nonlinear Control. The sufficient conditions for accomplishing the synchronization and AS of two identical hyperchaotic systems are derived based on Lypunov Stability Theory. Numerical simulations and graphs are furnished to show the effectiveness of our proposed approach. All simulations have been done by using*** **mathematica 9.**

Keywords— Synchronization, Lyapunov Stability Theory, Nonlinear Active Control, Hyperchaotic System

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1. INTRODUCTION

The idea of synchronization was first studied by L. Pacora and T. Carroll [1] which is based on designing a coupling between the two systems such that the chaotic time evaluation becomes ideal and the output of the slave (response) system asymptotically follows the output of the master (drive) system. Since then the synchronization of chaotic dynamical systems have received a great deal of interest among scientists from almost all nonlinear sciences for more than last two decades [2] and so far a large number of synchronization techniques have been developed and applied successfully to synchronize two identical as well as nonidentical chaotic systems [3]. Among them, chaos synchronization using Nonlinear Active Control techniques have recently been widely accepted as the most efficient techniques used for both synchronization as well as anti-synchronization of hyperchaotic systems as no gain matrix or Lyapunov exponents are required for its execution [4].

Hyperchaotic system is generally defined as a chaotic system which has at least two positive Lyapunov exponent and the presence of more than one positive lyapunov exponent clearly enhances the security by generating more complex dynamics which can be used to improve the capacity, efficiency and security of chaotic communication systems. Due to these characteristics, it has potential applications in different scientific fields [5-7].

The main objective of this paper is to study the global chaos synchronization and anti-synchronization of identical new hyperchaotic systems [8]. Based on Lyapunov stability theory [9] and using the approach in [4], a class of nonlinear control schemes will be designed to achieve the synchronization and anti-synchronization asymptotically globally. Numerical simulations and graphs will be furnished to show the effectiveness and advantages of our proposed approach.

The rest of the paper is organized as follows: in section 2, we give the problem statement and the proposed methodology. In section 3, we discuss the chaos synchronization and anti-synchronization of new identical hyperchaotic systems. In section 4, numerical simulations are presented to endorse the effectiveness of our method and in section 5, the concluding remarks are then given.

2. 2. DESIGNING OF A NONLINEAR ACTIVE CONTROLLER

Consider a drive and response systems arrangement for a chaotic system is described by the following differential equations:

$$\dot{x} = M_1 x + G_1(x) \quad \text{(Drive system)} \qquad (2.1)$$

$$\dot{y} = M_2 y + G_2(y) + \mu \quad \text{(Response system)} \qquad (2.2)$$

where **x, y** ϵ R^n are the corresponding state vectors, $M_1, M_2 \in R^n$ are the matrices (***n***×***n***) of system parameters and $G_1, G_2 : R^n \rightarrow R^n$ are the continuous nonlinear functions and ' $\mu(t)$ ' is a nonlinear state feedback controller yet to be design.

If $M_1 = M_2$ and/or $G_1(\cdot) = G_2(\cdot)$, then **x** and **y** are the states of two identical chaotic systems.

If $M_1 \neq M_2$ and/or $G_1(\cdot) \neq G_2(\cdot)$, then **x** and **y** are the states of two different chaotic systems.

The error dynamics for synchronization of chaotic systems (2.1) and (2.2) is described as,

$$\dot{e} = M_2 y - M_1 x + G_2(y) - G_1(x) + \mu(t) \tag{2.3}$$

where, $e_i = y_i - x_i$

For the two non-identical chaotic systems without controller, ($\mu_i(t) = 0$), if the initial conditions, $(x_{1i}(0), x_{2i}(0), ..., x_{ni}(0) \neq y_{1j}(0), y_{2j}(0), ..., y_{nj}(0))$, then the trajectories of the two chaotic systems will rapidly split from each other at the course of time and will become totally unsynchronized. Thus the synchronization problem is essentially to find a feedback controller ' $\mu(t) \in R^{n\times 1}$ ' such that, it stabilizes the error dynamics (2.3) for all initial conditions, i.e., if,

$$\lim_{t\to\infty} \| y_i(t) - x_i(t) \| = \lim_{t\to\infty} \| e_i(t) \| = 0, \quad \text{for all } e_i(0) \in R^n,$$

then the two chaotic systems (2.1) and (2.2) are said to be synchronized.

Let if we consider a candidate Lyapunov Error Function as:

$$V(t) = e^T A e$$

where the matrix **A** is a positive definite matrix [9]. It can be noticed that, $V : R^n \rightarrow R^n$ is a positive definite function by construction. We further assume that the parameters of the drive and response systems are known and the states of both chaotic systems are measurable.

We may achieve the synchronization by selecting a non-linear controller $\mu(t) \in R^{n\times 1}$ to make $-\dot{V}(e) = e^T B e$ be a positive definite matrix, then by Lyapunov Stability Theory [9], the states of the drive and response systems will be globally asymptotically synchronized.

3. SYNCHRONIZATION VIA NONLINEAR ACTIVE CONTROL

System Description: Liu Wen-Bo et. al., [8] proposed and studied a new continuous autonomous hyperchaotic system in which each equation in the system contains 2-term cross product and is described as:

$$\left.\begin{aligned} \dot{x} &= \kappa_1 x - yz \\ \dot{y} &= -\kappa_2 y + xz + \varphi w \\ \dot{z} &= -\kappa_3 z + xy \\ \dot{w} &= -\kappa_4 w + \xi xz \end{aligned}\right\} \tag{3.1.1}$$

where $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w} \in \boldsymbol{R}^n$ are the state variables and $\kappa_1, \kappa_2, \kappa_3, \kappa_4, \varphi$ and ξ are all positive real constant parameters with $\kappa_1 = 2, \kappa_2 = 3, \kappa_3 = 1.5, \kappa_4 = 2, \varphi \in [0.4, 3)$ and $\xi = -1$.

By linearization of system (3.1.1) at $(x, y, z, w) = (0, 0, 0, 0)$, the eigenvalues are: $\lambda_1 = 2$, $\lambda_2 = -3$, $\lambda_3 = -1.5$ and $\lambda_4 = -2$.

The chaotic system (3.1.1) exhibits a chaotic attractor when the parameter values are taken as: $\kappa_1 = 2$, $\kappa_2 = 3$, $\kappa_3 = 1.5$, $\kappa_4 = 2$, $\varphi = 1.6$ and $\xi = -1$.

For the dynamical properties such as hyperchaotic behavior, bifurcation, single scroll, the 2-scroll and 4-scroll chaotic attractors etc. for the system (3.1.1), please see reference [8].

To achieve the identical synchronization of the hyperchaotic system (3.1.1), let us consider the drive-response systems configuration is described as:

$$\left.\begin{aligned} \dot{x}_1 &= \kappa_1 x_1 - y_1 z_1 \\ \dot{y}_1 &= -\kappa_2 y_1 + x_1 z_1 + \varphi w_1 \\ \dot{z}_1 &= -\kappa_3 z_1 + x_1 y_1 \\ \dot{w}_1 &= -\kappa_4 w_1 - x_1 z_1 \end{aligned}\right\} \quad \text{(Drive system)} \qquad (3.1.2)$$

and

$$\left.\begin{aligned} \dot{x}_2 &= \kappa_1 x_2 - y_2 z_2 + \mu_1 \\ \dot{y}_2 &= -\kappa_2 y_2 + x_2 z_2 + \varphi w_2 + \mu_2 \\ \dot{z}_2 &= -\kappa_3 z_2 + x_2 y_2 + \mu_3 \\ \dot{w}_2 &= -\kappa_4 w_2 - x_2 z_2 + \mu_4 \end{aligned}\right\} \quad \text{(Response system)} \qquad (3.1.3)$$

where $\boldsymbol{x}_i, \boldsymbol{y}_i, \boldsymbol{z}_i, \boldsymbol{w}_i \in \boldsymbol{R}^n$ for i =1,2 are the state variables of the corresponding drive and response systems, κ_1, κ_2, κ_3, κ_4 and φ are system parameters and $\mu(t) = [\mu_1(\mathrm{t}), \mu_2(\mathrm{t}), \mu_3(\mathrm{t}), \mu_4(\mathrm{t})]^T \in R^{n\times 1}$ are the nonlinear controllers which is yet to be designed.

The aim of this section is to determine the nonlinear controller '$\mu(t) \in R^{n\times 1}$' such that the two identical hyperchaotic systems (3.2.2) and (3.2.3) are synchronize asymptotically globally.

The synchronization error dynamics of equations (3.2.2) and (3.2.3) is defined as,

$$\left.\begin{aligned} \dot{e}_1 &= \kappa_1 e_1 - (y_2 z_2 - y_1 z_1) + \mu_1 \\ \dot{e}_2 &= -\kappa_2 e_2 + \varphi e_4 + (x_2 z_2 - x_1 z_1) + \mu_2 \\ \dot{e}_3 &= -\kappa_3 e_3 + (x_2 y_2 - x_1 y_1) + \mu_3 \\ \dot{e}_4 &= -\kappa_4 e_4 - (x_2 z_2 - x_1 z_1) + \mu_4 \end{aligned}\right\} \qquad (3.1.4)$$

Thus the aim of this section is to synchronize two identical hyperchaotic systems (3.1.2) and (3.1.3) by designing such nonlinear feedback controllers that ensure the asymptotic stability of the error system (3.1.4). To achieve this goal, let us assume the following theorem.

Theorem 2. The two Chaotic Systems (3.1.2) and (3.1.3) will achieve asymptotically globally synchronization for all initial conditions $(x_1(0), x_2(0), x_3(0), x_4(0) \neq y_1(0), y_2(0), y_3(0), y_4(0))$ with following control law:

$$\left.\begin{aligned} \mu_1(t) &= -2\kappa_1 e_1 + (y_2 z_2 - y_1 z_1) \\ \mu_2(t) &= x_1 z_1 - x_2 z_2 \\ \mu_3(t) &= x_1 y_1 - x_2 y_2 \\ \mu_4(t) &= -\varphi e_4 + (x_2 z_2 - x_1 z_1) \end{aligned}\right\} \tag{3.1.5}$$

Proof. Let us assume that the states of both systems (3.1.2) and (3.1.3) are measurable and the parameters of the drive and response systems are known. Let us construct a Lyapunov error function candidate as,

$$V(t) = e^T A e \tag{3.1.6}$$

where $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ is a positive definite function.

Now the time derivative of the Lypunov Error Function is,

$$\dot{V}(t) = -2k_1 e_1{}^2 - 3k_2 e_2{}^2 - k_3 e_3{}^2 - 2k_4 e_4{}^2 = -e^T \begin{pmatrix} 2k_1 & 0 & 0 & 0 \\ 0 & 3k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & 2k_4 \end{pmatrix} e \prec 0$$

Therefore, $-\dot{V}(t) = e^T B e$ and $B = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 1.5 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$ which is also a positive definite matrix.

Hence based on Lyapunov Stability Theory [9], the origin of the error dynamics converge to the origin asymptotically. Thus drive and response systems (3.1.2) and (3.1.3) are asymptotically globally synchronized.

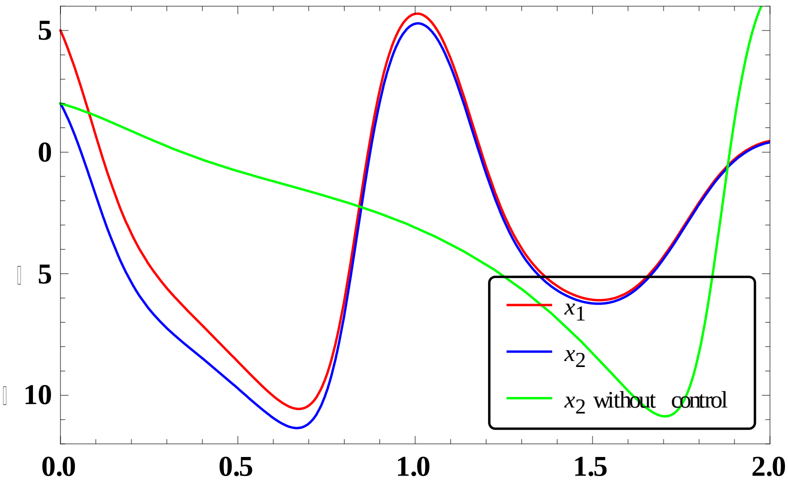

Fig 1: Time Series of x_1 & x_2 (During Synchrnization)

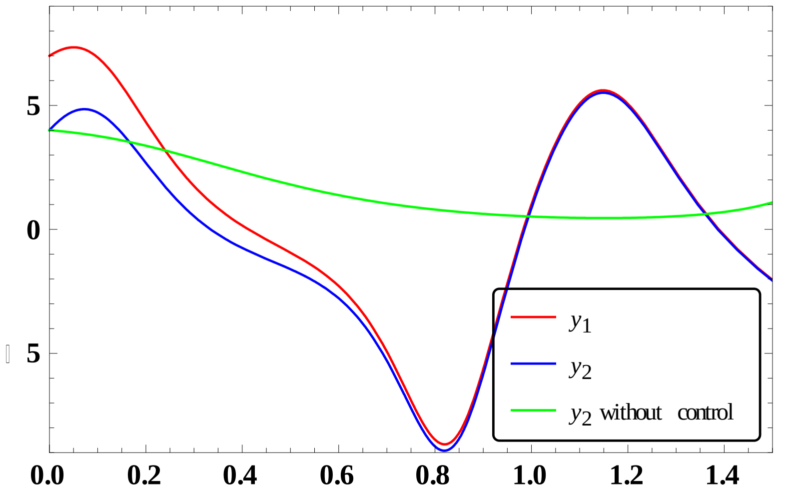

Fig 2: Time Series of y_1 & y_2 (During Synchrnization)

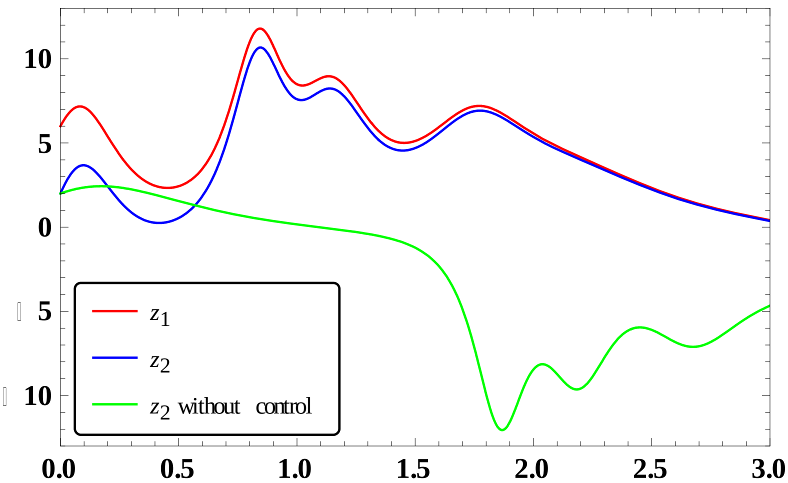

Fig 3: Time Series of z_1 & z_2 (During Synchrnization)

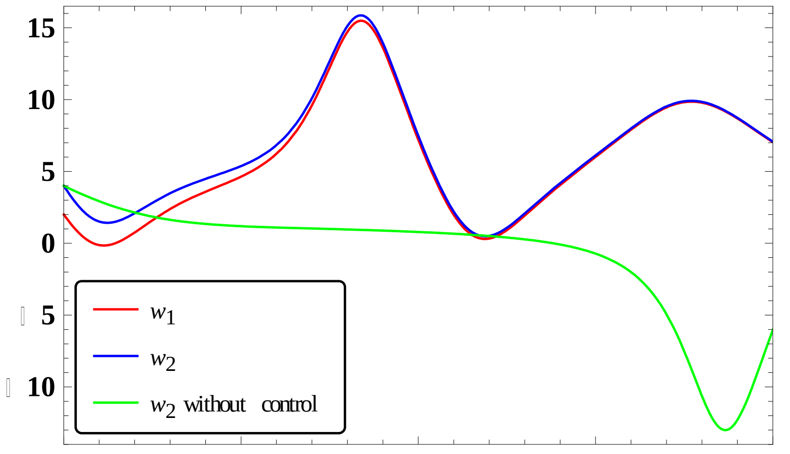

Fig 4: Time Series of w_1 & w_2 During Synchrnization

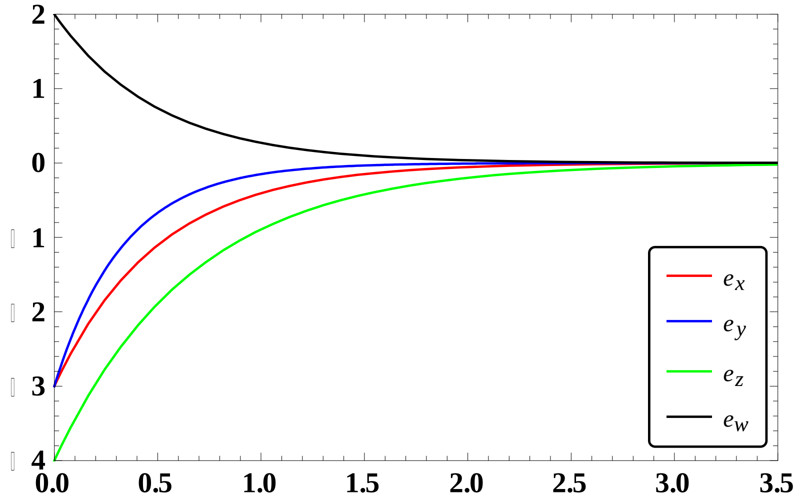

Fig 5: Time Series of errors During Synchronization

Anti-Synchronization via Nonlinear Active Control

In this section, we present the purpose of the study which is to achieve stable anti-synchronization between two identical hyperchaotic systems [8] by using Nonlinear Active Control. To achieve this goal, let us consider the drive-response systems arrangement for the identical anti-synchronization of a new hyperchaotic system [8] is described as:

$$\left.\begin{aligned} \dot{x}_1 &= \kappa_1 x_1 - y_1 z_1 \\ \dot{y}_1 &= -\kappa_2 y_1 + x_1 z_1 + \varphi w_1 \\ \dot{z}_1 &= -\kappa_3 z_1 + x_1 y_1 \\ \dot{w}_1 &= -\kappa_4 w_1 - x_1 z_1 \end{aligned}\right\} \quad \text{(Drive system)} \qquad (3.2.1)$$

and

$$\left.\begin{aligned}\dot{x}_2 &= \kappa_1 x_2 - y_2 z_2 + \mu_1\\ \dot{y}_2 &= -\kappa_2 y_2 + x_2 z_2 + \varphi w_2 + \mu_2\\ \dot{z}_2 &= -\kappa_3 z_2 + x_2 y_2 + \mu_3\\ \dot{w}_2 &= -\kappa_4 w_2 - x_2 z_2 + \mu_4\end{aligned}\right\} \quad \text{(Response System)} \qquad (3.2.2)$$

For chaotic Anti-Synchronization of the two drive-response systems, the error dynamics can be described as,

$$\left.\begin{aligned}e_1 &= x_1 + x_2\\ e_2 &= y_1 + y_2\\ e_3 &= z_1 + z_2\\ e_4 &= w_1 + w_2\end{aligned}\right\}$$

Thus from systems of equations (3.2.1) and (3.2.2), the error dynamics can be described as:

$$\left.\begin{aligned}\dot{e}_1 &= \kappa_1 e_1 - (y_1 z_1 + y_2 z_2) + \mu_1\\ \dot{e}_1 &= -\kappa_2 e_2 + \varphi e_4 + (x_2 z_2 + x_1 z_1) + \mu_2\\ \dot{e}_3 &= -\kappa_3 e_3 + (x_2 y_2 + x_1 y_1) + \mu_3\\ \dot{e}_4 &= -\kappa_4 e_4 - (x_2 z_2 + x_1 z_1) + \mu_4\end{aligned}\right\} \qquad (3.2.3)$$

Let us define the Nonlinear Controller $\mu(t) = [\mu_1(t), \mu_2(t), \mu_3(t), \mu_4(t)]^T \in R^{n\times 1}$ as,

$$\left.\begin{aligned}\mu_1(t) &= -2\kappa_1 e_1 + y_1 z_1 + y_2 z_2\\ \mu_2(t) &= -\varphi e_4 - (x_2 z_2 + x_1 z_1)\\ \mu_3(t) &= -(x_2 y_2 + x_1 y_1)\\ \mu_4(t) &= (x_2 z_2 + x_1 z_1)\end{aligned}\right\} \qquad (3.2.4)$$

Replacing equations (3.2.4) in (3.2.3), we have,

$$\left.\begin{aligned}\dot{e}_1 &= -\kappa_1 e_1\\ \dot{e}_1 &= -\kappa_2 e_2\\ \dot{e}_3 &= -\kappa_3 e_3\\ \dot{e}_4 &= -\kappa_4 e_4\end{aligned}\right\} \qquad (3.2.5)$$

Let us construct a Lyapunov Error Function Candidate as:

$$V(t) = \frac{1}{2}(e_1^{\,2} + e_2^{\,2} + e_3^{\,2} + e_4^{\,2}) \qquad (3.2.6)$$

Now the time derivative of the Lypunov Error Function is,

$$\dot{V}(t) = -k_1 e_1^{\,2} - k_2 e_2^{\,2} - k_3 e_3^{\,2} - k_4 e_4^{\,2} = -e^T \begin{pmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & k_4 \end{pmatrix} e \prec 0$$

Therefore, $-\dot{V}(t) = e^T Be$ and $B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1.5 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ which is also a positive definite matrix.

Hence based on Lyapunov Stability Theory [9], the origin of the error dynamics converge to the origin asymptotically. Thus drive and response systems (3.2.1) and (3.2.2) are asymptotically globally anti-synchronized.

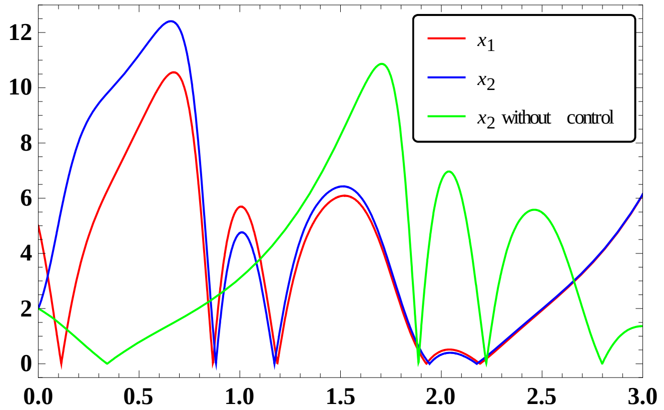

Fig 6: Time Series of x_1 & x_2 (During AS)

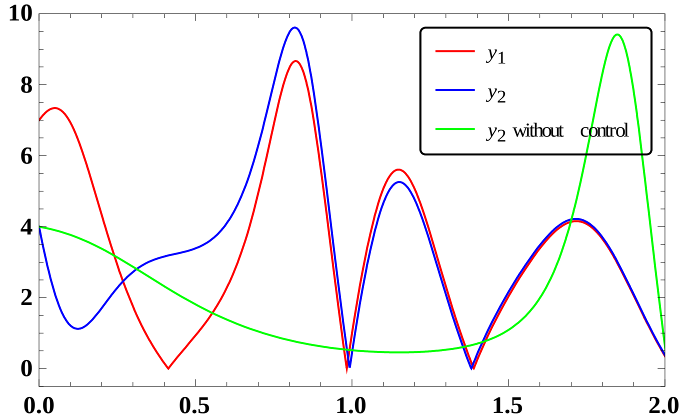

Fig 7: Time Series of y_1 & y_2 (During AS)

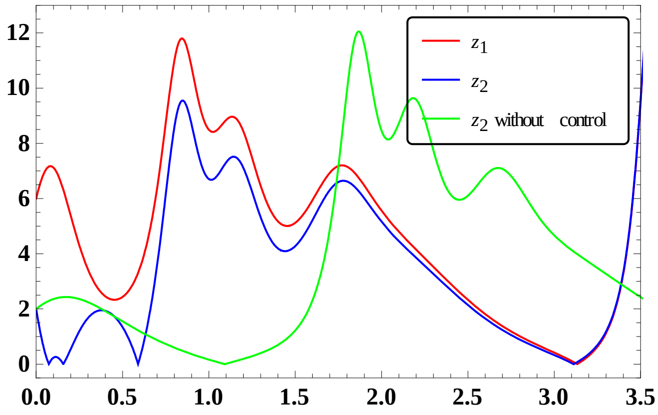

Fig 8: Time Series of z_1 & z_2 (During AS)

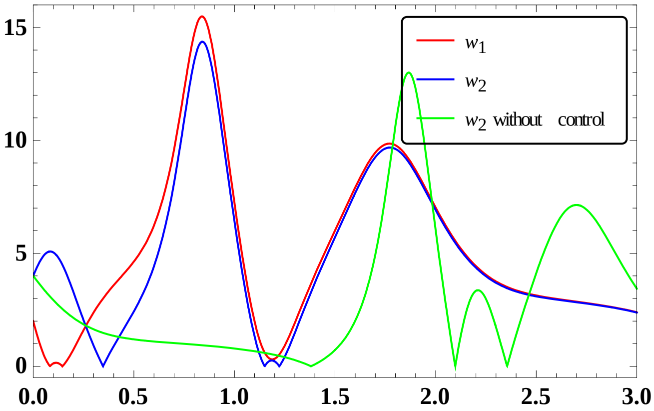

Fig 9: Time Series of w_1 & w_2 (During AS)

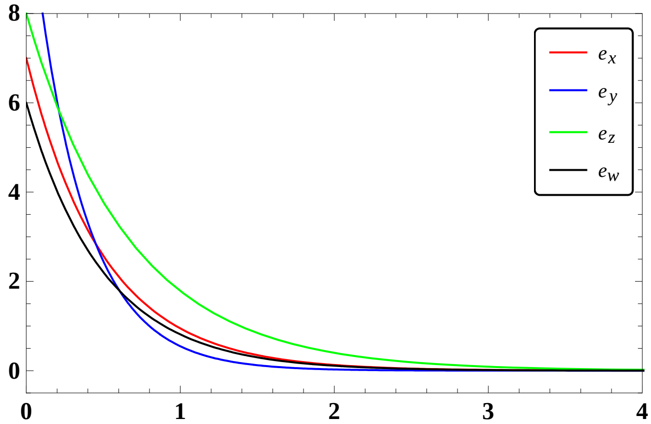

Fig 10: Time Series of errors (During AS)

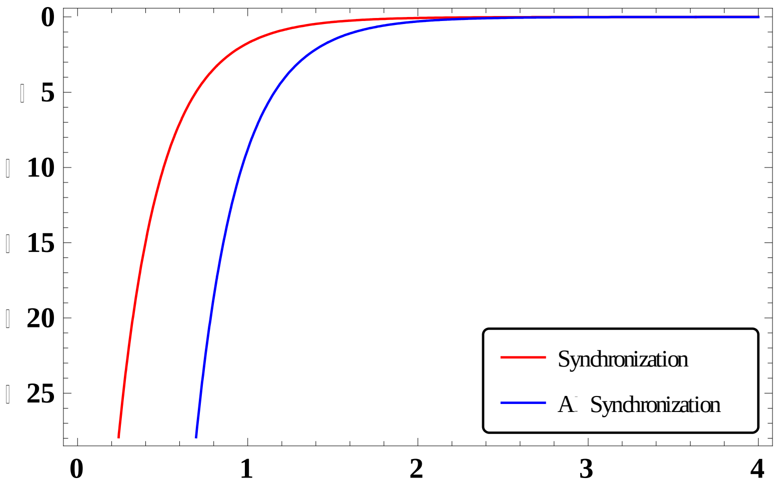

Fig 11: Time Series of V' t

4. NUMERICAL SIMULATIONS

For the hyperchaotic System [8], the parameter values are taken as $\kappa_1 = 2$, $\kappa_2 = 3$, $\kappa_3 = 1.5$, $\kappa_4 = 2$, $\varphi = 1.6$ and $\xi = -1$ together with initial conditions: $(x_1(0), y_1(0), z_1(0), w_1(0)) = (5, 7, 6, 2)$ and $(x_2(0), y_2(0), z_2(0), w_2(0)) = (2, 4, 2, 4)$, we have plotted the time series of states variables for synchronization (Fig 1-4) as well as for anti-synchronization (Fig 6-9) whereas figures 5 & 10 depict the time series of the errors for synchronization & anti-synchronization respectively. Lastly, in fig 11, we have drawn the graphs of the derivatives of the Lyapunov Errors Functions for both the cases in order to show the stability.

5. SUMMARY AND CONCLUSION

In this paper, global chaos synchronization and anti-synchronization of an identical new hyperchaotic system have been studied. Based on Lyapunov Stability Theory and using the Nonlinear Active Control technique, a class of nonlinear controllers is designed to achieve the global stability of the error dynamics. Since the Lyapunov exponents or gain matrix are not required for numerical simulations, the Nonlinear Active Control Technique is an effective algorithm to synchronized and anti-synchronized two identical hyperchaotic systems.

In this study, using the Nonlinear Active Control Technique, it has been shown that the proposed approaches have exceptional transient performances and that analytically as well as graphically the synchronization and anti-synchronization are asymptotically globally stable. In this study, it can be observed that synchronization is working faster than anti-synchronization. Results are presented in graphical forms with time history (figures 1-11). Numerical simulations are also given to validate the results. Figures 5 shows the synchronization errors figure 10 shows the anti-synchronization errors for two identical hyperchaotic systems [8] respectively which shows that the proposed controllers are efficient with enough transient speed. Figure 11 show the derivatives of Lyapunov Error functions of identical synchronization and identical anti-synchronization respectively of the systems which shows that the error systems (figures 5 and 10) are feedback stabilized. Numerical simulations are also given to validate the results.

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