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Performance Evaluation of 2^{nd} Order Localized Approximation in Predicting Internal Electric Field in One-Dimensional Slab Problem

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Abstract

We investigate the performance of a new 2^{nd} order localized approximation, as extracted from the generalization of the localized approximation. These 2^{nd} order localized approximation is then applied in the study of electromagnetic field interaction of plane waves with a slab of complex conductivity. The approximation allowed us to obtain explicit expressions for the internal electric field of the one dimensional slab from the integral equations for a piece-wise constant and linear profiles. These equations were employed to compute the internal electric fields within the slab support. The computed results were compared with the exact solutions obtained from transmission-line theory. In addition we iteratively solved for the internal electric field from the intergral equations. The three solutions were then evaluated and compared at two distinct frequencies. The effect of background profile and contract between the slab and background profiles on the accuracy of the approximation were thoroughly investigated in detail. We also derived explicit expressions for the internal electric field for a slab with linear complex conductivity profile. However, these were never numerically computed and evaluated. We defined as L^2 error norm, with reference to the exact solution from Transmission-line theory as a metric for the performance evaluation of the computed internal fields. The results are presented in both graphical and table forms. The Original Habashy approximation solutions were compared against the new 2^{nd} order solutions. We conclude with evaluation of the performances of the approximations and recommendations for the next logical direction in future investigation of the localized approximations.

1 Introduction

In electromagnetic inverse problems, we desire to extract accurate information about the complex conductivity or permittivity profile composition of a medium from its measured interaction with electromagnetic radiation. This is not an easy problem and its computational difficulties are well documented [3, 4], [2]−[5]. Over the years attempts have been made to perfect this art of extracting useful profile information from imperfectly measured electromagnetic scattering data [7, 8, 9, 11]. The essential first step to achieve this objective is finding an accurate forward model for the scattering fields of the electromagnetic scattered fields.

In the Green function formulation of the electromagnetic scattering problem, this reduces to find a simple and feasible method for computing the internal scattered fields within the medium. A number of techniques have been developed to solve the Green function method integral equation. The main difficulty with the Green function method is that, the integrand is product of the profile function and the internal field which are mutually dependent. The implication is that, the integral equation is extremely nonlinear. The earliest attempts at solving integral equations of these kinds were Born[6, 12, 15] and Rytov[14] and by Habashy et al [10]. In recent times, Habashy and other [7] proposed the Localized Approximation technique for solving the internal electromagnetic electric fields of the integral equation.

The localized approximation technique is based on the observed localization properties of the Green's function in higher dimensional problems and the smoothly varying internal electromagnetic fields. In a follow up work to those of Habashy et al [7], Adopley [1] extended the localized approximation to the one dimension (1D) case, where the singularity of the Green's function in higher dimensions degrades into a localized peak for complex profile systems. Adopley [1] used the localized approximation technique to investigate the performance of the first order Habashy approximation, and a 2^{nd} order Adopley approximations in a 1D slab-problem. However in that work nothing was done to study the performance of an alternative 2^{nd} order approximation based on the second of the two generalized localized approximation model developed in [1, 7]. The performance of this alternative 2^{nd} order localized approximation, is the main focus of this work. We study in great details its potential in accurately predicting the internal field of the integral equation. The investigation is limited to the 1D slab with piece-wise constant complex conductivity profile as used in [11]. However we present the Green function formulation in all three dimensions. The Green function formulations are presented just for completeness. These derivations are readily available [8, 9, 16]. It is instructional to note the fact that, the Green function of the 1D slab-problem has the least localizing power with respect to higher dimensions. In addition the exact internal electric fields are readily computed from Transmission-Line theory (TL-Mode) [13] for the piece-wise constant profile. It is noted that, apart from the distinct expressions for the Green function in each dimension, the general format of the electromagnetic integral equation are identical in all dimensions.

In the next `section`, we define the geometry of the slab problem with its governing Green function integral equation. In addition we present the Green function integral equations in higher dimensions (3D and 2D), with explicit expressions for the respective Green functions. The following `section`, provides analytic expressions for the internal electric field of the piece-wise constant profile of the 2^{nd} order localized approximation. `Section` 4 provides the explicit solutions for the linear slab profile. However the derived expressions for the linear profile were never applied in any computations. The investigation of their performance is left for the future.

In `section` 5 we present the solution models and provide extensive numerical study of the performance of the second order localized approximation against the Habashy approximation at two frequencies of 2.4 MHz and 2.4 GHz. The 2^{nd} Order Localized Approximation numerical performance were then compared against the Habashy Localized Approximation and a direct numerical iterative solution of the integral equation. All approximate solutions were evaluated against the exact solution from transmission-line theory. Deviations of the approximate solutions from the exact solution are displayed both in tables and graphs. We concluded with our findings on the performance of the second order approximation and provided our recommendation and the directions of our future work in the last section.

2 Slab problem and its Integral Equations

The slab problem geometry is displayed in Figure 1. The integral equation governing interaction of plane wave with the slab was derived in [1] and is given by

$$E_y(z) = E_y^{in}(z) + \frac{k_b}{2\jmath\bar{\epsilon}_b}\int_0^d \bar{q}(z^{'})E_y(z^{'})g(z,z^{'})dz^{'} \tag{1}$$

in terms of complex permittivity or

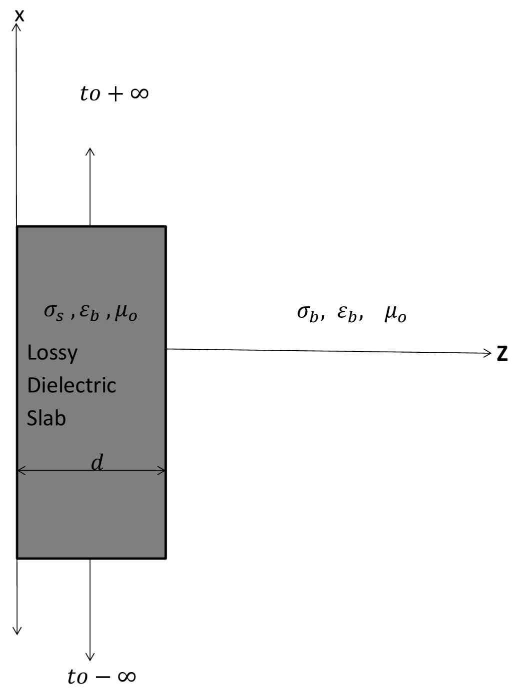

Figure 1: Lossy Dielectric slab

$$E_y(z) = E_y^{in}(z) + \frac{k_b}{2\jmath\bar{\sigma}_b}\int_0^d \bar{\sigma}(z^{'})E_y(z^{'})g(z,z^{'})dz^{'} \tag{2}$$

in terms of complex conductivity. Here $\bar{q}(z)$ and $\bar{\sigma}(z)$ are complex permittivity and complex conductivity contrast respectively. $\bar{q}(z)$ is defined as $\bar{\epsilon_s} - \bar{\epsilon_b}$ and relative complex permittivity is defined as $\bar{\epsilon} = \epsilon_r + \jmath\sigma/(\omega_0\epsilon_0)$. Similarly complex conductivity is defined as $\bar{\sigma} = \sigma - \jmath\omega\epsilon_r\epsilon_0$. It is instructional to know that, the integral formulation of electromagnetic scattering problems with Green's function techniques assume the same form in multiple dimensions. In the cases of 2D and 3D systems, the integral equations assume the form [9]

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_b(\mathbf{r}) + \int_\tau \Delta\,\bar{\bar{\sigma}}(\mathbf{r}_0)\,\bar{\bar{G}}(\mathbf{r},\mathbf{r}_0)\,\mathbf{E}(\mathbf{r}_0)\,d\mathbf{r}_0 \tag{3}$$

where $\bar{\bar{G}}(\mathbf{r},\mathbf{r}_0)$ is the Dyadic Green's function. The differential equations that govern the Green functions and their solutions are well presented in the works of Gao et al [9, 16].

In the works of Adopley et al [11] on localized approximation of the slab problem, nothing was said about the performance of the 2^{nd} order of the second type generalization of the localized approximation. Specifically, the approximation of the second type generalization, specialized to the 2^{nd} order gives the internal field as

$$E^{int}(z) = \Gamma_N(z)\cdot\sum_{n=0}^{N-1} E_n(z) \tag{4}$$

and

$$\Gamma_N(z) = [1-(\Omega(z))^N]^{-1}. \tag{5}$$

with

$$\Omega(z) = \frac{k_b}{2\jmath\bar{\sigma}_b}\int_0^d \bar{\sigma}(z^{'})g(z,z^{'})dz^{'} \tag{6}$$

$E_n(z)$ is defined recursively as

$$E_n(z) = \frac{k_b}{2\jmath\bar{\sigma}_b}\int_0^d E_{n-1}(z^{'})\bar{\sigma}(z^{'})g(z,z^{'})dz^{'} \tag{7}$$

and $E_0(z) = E_b(z)$ is the incident field. The 2^{nd} order localized approximation of the second approximation then becomes, for the internal electric field

$$E^{int}(z) = \frac{1}{(1-[\Omega(z)]^2)}\cdot\left[E_b(z) + \frac{k_b}{2\jmath\bar{\sigma}_b}\int_0^d \bar{\sigma}(z^{'})E_b(z^{'})g(z,z^{'})dz^{'}\right] \tag{8}$$

The accuracy of Equation 8 in predicting the internal electric field is our main focus in this work. Armed with the knowledge of the complex conductivity distribution of the slab, the internal electric field can be ready evaluated using the above approximation. In this study on the performance of the 2^{nd} order approximation, we generated a generic analytical solution of the integral equation for the internal field in case of the piece-wise constant profile. This should not be regarded as a set back as any complex profile distribution, can be reduced to piece-wise constant profile approximation to any desired accuracy. Indeed, with computing power presently at the disposal of researchers and application of parallel computing, this may be the best available option as individual discretized integration components are independent of each other and hence can all be evaluated in parallel. The above expressions for the internal electric field will now be solved for piece-wise constant and linear profiles explicitly.

3 2^{nd} Order Localized Approximation: Piece-wise Constant Profile Solution

We need to compute the integrals in the expressions for both $\Omega(z)$ and $E_n(z)$ in order to evaluate the internal electric field. Note that for the 2^{nd} Order approximation where $N = 2$, the two integrals that we need to evaluate are

$$\Omega(z) = \frac{k_b}{2\jmath\bar{\sigma}_b}\int_0^d \bar{\sigma}(z')g(z,z')dz' \tag{9}$$

and $E_{in}(z) = E_0(z) + E_1(z)$ is given by

$$E_{in}(z) = E_b(z) + \frac{k_b}{2\jmath\bar{\sigma}_b}\int_0^d E_b(z')\bar{\sigma}(z')g(z,z')dz' \tag{10}$$

Note that the expression for the Green function for the 1D case is given by $g(z,z') = e^{-\jmath k_b|z-z'|}$ and $E_b(z) = e^{-\jmath k_b z}$ for the incident plane wave. For a piece-wise constant profile, the two integrals are readily evaluated as

$$\begin{aligned}\Omega(z) &= \frac{k_b}{2\bar{\sigma}_b}\left(-\left[\sum_{n=1}^{m-1}\bar{\sigma}_n\left(e^{\jmath k_b z_n} - e^{\jmath k_b z_{n-1}}\right) - \bar{\sigma}_m e^{\jmath k_b z_{m-1}}\right]e^{-\jmath k_b z}\right.\\ &+ \left.\left[\sum_{n=m+1}^{N}\bar{\sigma}_n\left(e^{-\jmath k_b z_n} - e^{-\jmath k_b z_{n-1}}\right) - \bar{\sigma}_m e^{-\jmath k_b z_m}\right]e^{\jmath k_b z} - 2\bar{\sigma}_m\right)\end{aligned} \tag{11}$$

and

$$\begin{aligned} E_2(z) &= \frac{e^{\jmath k_b z}}{4\bar{\sigma}_b}\left(\left[e^{-2\jmath k_b z_m} - e^{-2\jmath k_b z}\right] + \sum_{n=m+1}^{n}\bar{\sigma}_n\left[e^{-2\jmath k_b z_n} - e^{-2\jmath k_b z_{n-1}}\right]\right) \\ &+ e^{-\jmath k_b z}\left(1 + \frac{k_b}{2\jmath\bar{\sigma}_b}\left[\sum_{n=1}^{m-1}(z_n - z_{n-1})\bar{\sigma}_n + \bar{\sigma}_m(z - z_{m-1})\right]\right) \end{aligned} \tag{12}$$

Equations 11 and 12 are then deployed to investigate the performance of the 2^{nd} order localized approximation against the exact solution from transmission-line theory, the Habashy approximation and a directly iterative solution of the integral equation. In the case of the iteration of the integral equation, we assume the initial solution $E_y^0(z)$ and then apply the iteration step

$$E_y^n(z) = E_y^{in}(z) + \frac{k_b}{2\jmath\bar{\sigma}_b}\int_0^d \bar{\sigma}(z^{'})E_y^{n-1}(z^{'})g(z,z^{'})dz^{'} \tag{13}$$

to numerically compute an approximation for E_y.

4 2^{nd} Order Localized Approximation: Linear Profile Solution

In this section we developed the analytical solutions for the linear profile for the 2^{nd} Order localized approximation. It should be noted that, for more complex geometries, appropriate fundamental basis functions can be used to represent the profile. In the future, Fourier series analysis will be employed to model the complex conductivity profiles with compact support. Presently the linear profile is used to display the power of the localized approximation technique as the relevant integral can be expressed in close form.

For the linear profile, we assume the complex conductivity distribution to be of the form $\bar{\sigma}(z) = \bar{\alpha}z + \bar{\sigma}_0$, where $\bar{\sigma}_0$ is some constant complex conductivity and $\bar{\alpha}$ is some complex number constant. The solution can then be separated into two parts using linearity of the integral equation. The first part due to the constant term, is what was already analyzed in the piece-wise constant profile. The second part due the z dependent part is what needs to be evaluated. Since $\bar{\alpha}$ is just a complex number constant that scales through, it not really needed. Hence, the linear form analyzed is simply $\bar{\sigma}(z) = z$. The two integrals to evaluate are then

$$\Omega(z) = \frac{k_b}{2\jmath\bar{\sigma}_b} \int_0^d z^{'} g(z, z^{'}) dz^{'} \tag{14}$$

which evaluates to

$$\Omega(z) = \frac{1}{2\jmath k_b \bar{\sigma}_b} \left[(1 + \jmath k_b\, d)\, e^{-\jmath k_b (d-z)} - e^{-\jmath k_b z} - 2\jmath k_b\, z \right] \tag{15}$$

and $E_{in}(z) = E_0(z) + E_1(z)$ is given for the linear profile by

$$E_{in}(z) = E_b(z) + \frac{k_b}{2\jmath\bar{\sigma}_b} \int_0^d E_b(z^{'}) z^{'} g(z, z^{'}) dz^{'} \tag{16}$$

which evaluates to

$$\begin{aligned} E_{in}(z) &= \left(1 + \frac{k_b z^2}{4\jmath\bar{\sigma}_b}\right) e^{-\jmath k_b z} \\ &+ \frac{e^{\jmath k_b z}}{8\jmath k_b \bar{\sigma}_b} \left[(1 + 2\jmath k_b d)\, e^{-2\jmath k_b d} - (1 + 2\jmath k_b z)\, e^{-2\jmath k_b z} \right] \end{aligned} \tag{17}$$

5 Numerical Performance Evaluation

We computed the internal electric field of the slab problem using expressions derived for 1^{st} and 2^{nd} Order Habashy approximations, the iterative method, and the exact solution from Transmission-line theory for uniform complex conductivity profile. When there is no contrast between the background and slab complex conductivity profiles, all approximations produced the same results. The numerical computers were performed at two frequencies. A lower frequency of 2.4 MHz and a higher frequency of 2.4 GHz were used for the analysis evaluation. In this analysis we did not include the **Adopley Localized Approximation** since its performance against the 1^{st} Order Habashy approximation is well documented in [1]. We used background conductivity of 1.0 s/m ($\sigma_b = 1.0$) and relative permittivity of 1 ($\epsilon_b = 1.0$) throughout the analysis. In our numerical experimentation, we first varied the slab conductivity (σ_s) from $1.0 - 1.5$ s/m in step of 0.1 with constant slab permittivity at the frequency of 2.4 GHz. At the lower frequency, the slab conductivity was varied from $1 - 2.0$ in steps of 0.2. Then in our second computation, we swept the slab permittivity (ϵ_s) from $1.0 - 4.0$ in steps 0.3 also with constant background complex conductivity profiles for both frequencies of 2.4

MHz and 2.4 GHz. As the contrast in complex conductivity profile between the background and slab increases, the approximations degrade in numerical performance with the degree of degradation tracking the profile contrast magnitude.

The first set of plots, Figure 2 are for slab profile of $\sigma_s = 1$ and $\epsilon_s = 4.0$ The plots displays the Magnitude and Phase of computed internal electric field at 2.4 MHz. The second set of plots in Figure 3 display the same information for the same slab profile at 2.4 GHz. From the plots we note that all approximations accurately modelled the internal E-field at 2.4 MHz. The 2^{nd} Order Habashy model providing the best results at 2.4 GHz.

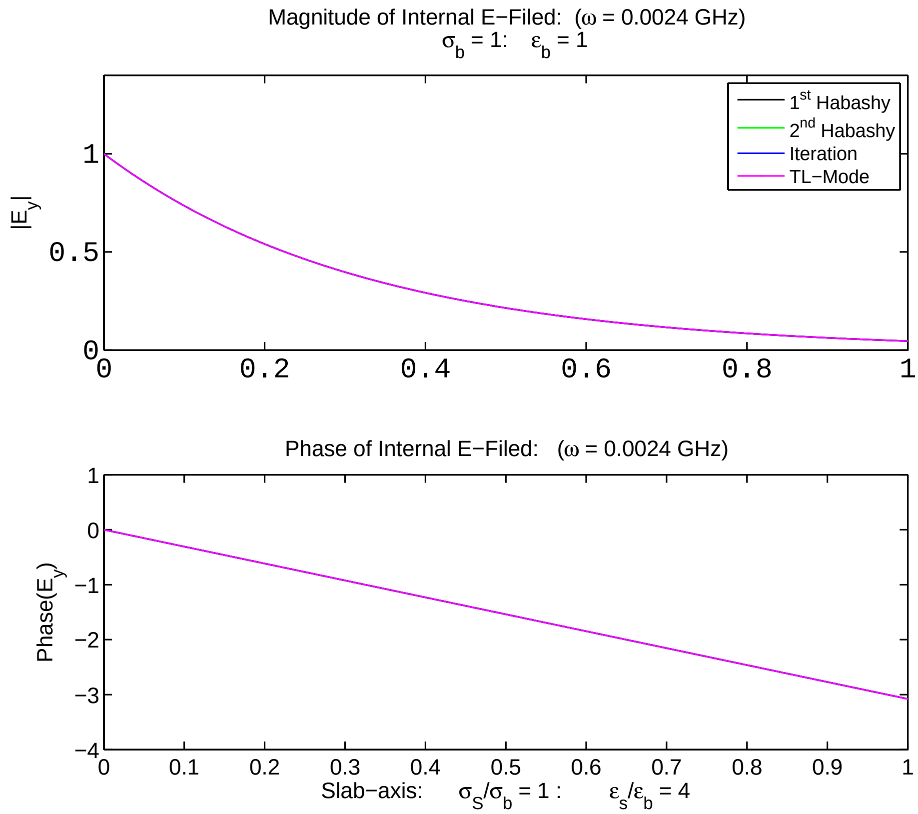

Figure 2: Manitude and Phase of Internal E-Field.

In the performance evaluation we defined an L^2-Error norm to quantify the performance metric of the approximation. We define an L^2-Error norm

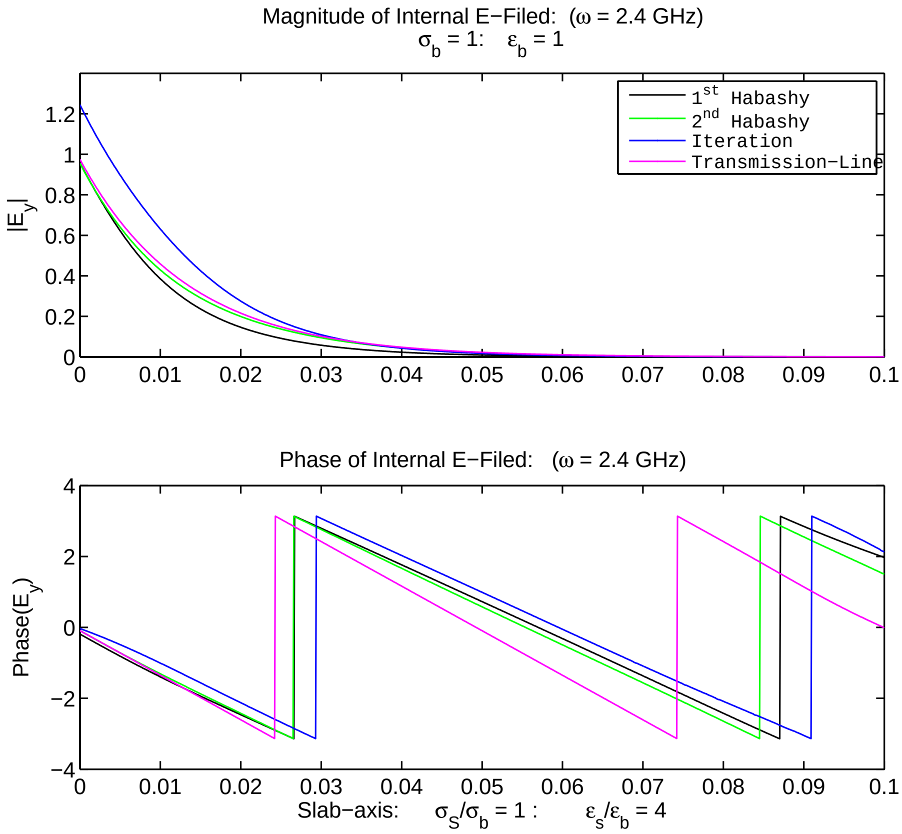

Figure 3: Magnitude and Phase of Internal E-Field.

as

$$||\delta E_y{}^{err}|| = \sqrt{\left(\frac{1}{d}\int_0^d \left[E_y(z) - E_y^{TL}(z)\right]^2 dz\right)} \tag{18}$$

based on difference between the internal electric fields of the approximations and the TL-Mode values. The error metric as computed by equation 18 are displayed in Table 1 and Table 2 at 2.4 MHz and 2.4 GHz respectively for the plots in Figures 2 and 3.

In our next set of plots, Figure 4 and 5 show results for the same background profile of unit relative permittivity and unit conductivity at 2.4MHz and 2.4 GHz respectively. The slab relative permittivity in these computations is kept at unity. We used conductivity of 2.0(s/m) at 2.4MHz and

Approximation Mode	$\|\|\delta E_y^{er}\|\|$
1^{st} Order Habashy	3.2048e-005
2^{st} Order Habashy	2.2137e-006
Iteration	1.2525e-004

Table 1: $F = 2.4$ MHz.; $\sigma_b = 1$; $\sigma_s = 1$; $\epsilon_b = 1$; $\epsilon_s = 4$

Approximation Mode	$\|\|\delta E_y^{er}\|\|$
1^{st} Order Habashy	0.0153
2^{st} Order Habashy	0.0054
Iteration	0.0871

Table 2: : $F = 2.4$ GHz.; $\sigma_b = 1$; $\sigma_s = 1$; $\epsilon_b = 1$; $\epsilon_s = 4$

Approximation Mode	$\|\|\delta E_y^{err}\|\|$
1^{st} Order Habashy	0.0394
2^{st} Order Habashy	0.1254
Iteration	0.1578

Table 3: $F = 2.4$ MHz.; $\sigma_b = 1$; $\sigma_s = 2.0$; $\epsilon_b = 1$; $\epsilon_s = 1$

Approximation Mode	$\|\|\delta E_y^{err}\|\|$
1^{st} Order Habashy	0.0145
2^{st} Order Habashy	0.0096
Iteration	0.0568

Table 4: $F = 2.4$ GHz.; $\sigma_b = 1$; $\sigma_s = 1.5$; $\epsilon_b = 1$; $\epsilon_s = 1$

Approximation Mode	$\|\|\delta E_y^{err}\|\|$
1^{st} Order Habashy	0.0071
2^{st} Order Habashy	0.0015
Iteration	0.0281

Table 5: $F = 2.4$ GHz.; $\sigma_b = 1$; $\sigma_s = 1.2$; $\epsilon_b = 1$; $\epsilon_s = 1$

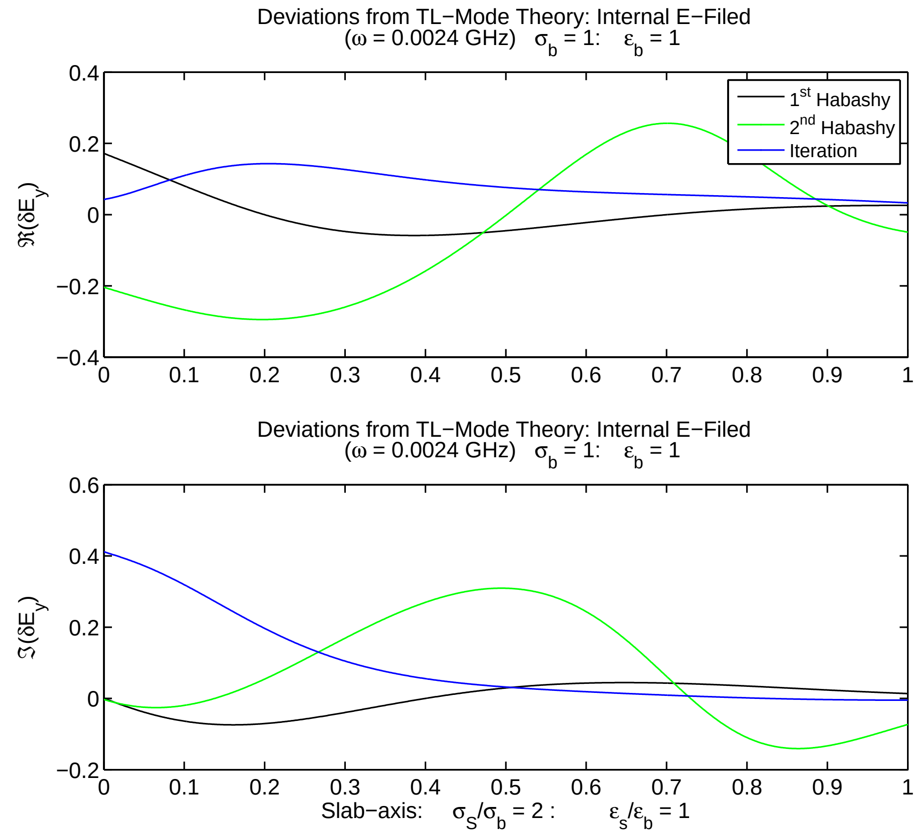

Figure 4: Relative Error to TL-Mode: Real & Imaginary Components.

1.5(s/m) at 2.4GHz for the slab. The plots of $\delta E_y(z)$ are defined as the difference between, internal E-filed approximations and the TL-Mode. We note from Figure 5 the relative superior performance for the 2^{nd} Order Habashy model. The iterative method provides the least accurate results relative to the TL-Mode. The relative error information are provided in Table 4. In our numerical experimentation, we uncovered The relative error information are provided in Table 4. In our numerical experimentation, we uncovered some subtle results. We noted that, at 2.4 MHz, the 1^{st} Order Habashy model outperforms the 2^{nd} Order Habashy model with better accuracy as the conductivity contrast increases between the background and the slab. Table 3 provides the $L^2 Error$ norm as defined by equation 18. It was also noted that, above conductivity contrast of 1.5 with the applied background profile, the

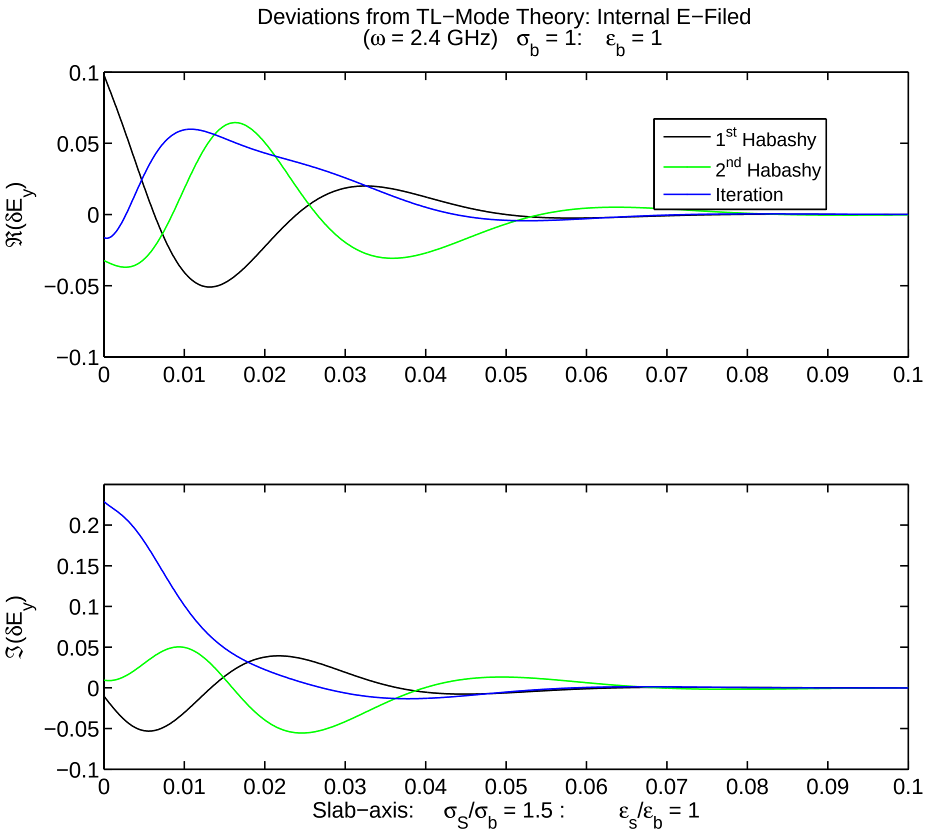

Figure 5: Relative Error to TL-Mode: Real & Imaginary Components.

Iteration Model provides better results than both Habashy models. We note in Table 5 that, at the high frequency of 2.4 GHz, the accuracy improves for all approximations with lower conductivity contrasts.

6 Conclusion

With the 2^{nd} Order localized approximation extracted from the generalization of the extended Born approximation, we obtained closed-form explicit solutions of the 1D slab problem for both piece-wise constant and linear profiles. From the extensive numerical computations, we succeeded in comparing the performance of the 2^{nd} localized approximation with the Habashy approximation, the direct iterative solution of the 1D integral equation and the exact solutions from TL-theory at two different frequencies of 2.4 MHz and 2.4 GHz. At the lower frequency, all approximations were very accurate with respect to the magnitude of phase data of the internal electric field computations relative to the exact solution from transmission-line theory for a slab permittivity contrast of ≤ 4 and conductivity contrast of 1. In all computations, the background profile was set at $\epsilon_b = 1$ and $\sigma_b = 1$ (s/m). We defined an L^2 Error norm which was used as the metric to evaluate the performance of the approximations. However, restricting the permittivity contrast to unity, and increasing the conductivity contrast of the slab against the background, the Habashy approximation tends to simulate the internal electric fields better than the 2^{nd} order approximation with increasing conductivity contrast at the 2.4 MHz range. In a follow up work, we intend to the adopt the 2^{nd} localized approximation as a forward model for inverse profile reconstruction.

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