

Application of an Innovate Energy Balance to Investigate Viscoelastic Problems

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ABSTRACT— *Modeling and investigating of energy distribution especially the wasted one is very important in viscoelastic problems. In this article, an applied energy model based on separation of energy components of the system is extracted and expanded to apply in linear viscoelastic problems, although this method is applicable in nonlinear problems as well. It is assumed that the whole energy of the system can be divided into two parts: Residual and non-inertial energies. The non-inertial energy is the sum of the energies that do not depend on the inertia of the system, while residual energy is the remaining of total energy. When an amount of energy is applied to the system, by determining the non-inertial energy from a novel energy conservation equation, the residual energy can be calculated. Some basic viscoelastic examples are investigated and obtained results will be compared with the expected ones.*

Keywords— Energy Balance; Residual Energy; Non-inertial energy; Viscoelasticity

1. INTRODUCTION

Energy dissipation in solids can be occurred by several different mechanisms, and although ultimately these all result in the mechanical energy being transformed into heat, two main dissipative processes are involved. The first type is known as 'static hysteresis'; that means the energy loss per cycle is independent from frequency, where the main cause may be associated simply with the 'static' non-linear stress-strain behavior of the materials. Another type is known as 'viscosity' properties of the body, according to which many materials show losses that are related to the velocity gradients set up by the vibrations. The forces producing these losses may be considered to have a viscous nature and mention that the mechanical behavior will depend upon the rate of strain. This issue is the subject of the 'linear viscoelasticity' [1].

When some energy is applied to a viscoelastic body, a part of the applied energy is dissipated as heat because of viscosity properties of matter, but the remaining is stored as reversible energy than is known as potential energy. It is frequently of interest to determine, for a viscoelasticity body at a given mode of displacement or deformation, the whole energy of deformation as well as the amount of energy stored or dissipated that can be investigated by studying the structure of viscose material [2,3,4].

The total deformational energy can be stated as follows:

$$W(t) = W_s(t) + W_d(t) \quad (1)$$

Where $W_s(t)$ is the energy stored and $W_d(t)$ is the energy dissipated. Equation (1) can be rewritten as follows:

$$\dot{W}(t) = \dot{W}_s(t) + \dot{W}_d(t) \quad (2)$$

Where all terms will refer to the unit volume of the body. Equation (2) may be looked upon as the definition of a viscoelastic material [4]. Based on this definition, a viscoelastic material is one in which the total applied energy is partly stored elastically and remaining dissipated as heat. Since applied energy to the body can only be stored or dissipated, Eq. (1) is quite generally true and also all terms are positive. What makes it a viscoelastic relation is the interpretation of the stored energy as purely potential energy. The precise form of Eq. (1) depends, of course, on the nature of the material on the one hand, and on the mode of the displacement or deformation on the other. It should be noted that equation (1) is not limited to investigating the linear behavior of viscous matters. The behavior studied here does not depend on the assumption that the response can be modeled by springs and dashpots as mechanical model. It may be simply supposed of energy storing and dissipating mechanisms without recognizing them with mechanical models, and modify the proof as needed as

an energy studying. For energy studying, it is necessary to know enough information on the applied deformation and the rate of the applied energy [5,6,7,8].

Reversible stored energy is known as elastically or potential energy. Energy may also be stored inertially, called here as residual energy, which may be encountered in fast loading experiments such as response to impulsive excitation, or in wave propagation at high frequency. Although in the linear theory of viscoelasticity, inertial energy storage plays no role in investigating the behavior of the matter, one may wish to calculate it [4]. The deformation of these structures requires exchange of residual energy and various internal elastic energies. This energy exchange is realized through special coupling of the transport of the internal elastic variables and the induced elastic stress which can be determined by using of the Hamilton's principle [9,10]. Some researchers have studied the structure and energy dissipation by using of other methods [11,12,13]. Therefore, general analysis of equation (1) and residual energy of body as well as energy dissipated or stored will be frequently of interest.

In this paper, by presenting a novel glance to energy components and their balance, the residual energy is defined and calculated as well as non-inertial energy which is the sum of the all energies that do not depend on the inertia of the system. Also, the presented approach is expanded to different basic linear viscoelasticity models and the results are discussed.

2. MATHEMATICAL MODELING

The certain effects of viscoelastic material behavior manifest themselves in a time-dependent response to loading and accompanying energy dissipation. The response of these materials will depend on the amount and rate of applied energy to the body. In fact, it can be stated that the relaxation or creep occurs when a viscoelastic matter is exposed to quasi-static displacement or loads and their changes. These phenomena are treated typically in the time and in the frequency domain, respectively [14].

By examining the activated energy components in the performed process, the non-inertial energy can be investigated, and thus, by applying the principle of energy conservation, the inertia-dependent energies of the body, hereafter called residual energy, are calculated. The presented approach can provide a practical idea for analyzing the relevant problems.

It is assumed that the total energy of the system can be stated as follows:

$$U_T = U_V + U_R \tag{3}$$

Where U_T is total energy, U_V is non-inertial energy and U_R is residual energy.

From the perspective of non-inertial energies, when an amount of energy is applied to the system, some of the energy components will be activated. Between these activated components, some of them change independently and changing of other components will be dependent on the independent energies. In this paper the bellow equation is used for non-inertial energies:

$$U_V = U_{V_s} + U_{V_d} \tag{4}$$

$$U_{V_s} = (u_1 + u_2 + \dots + u_m) + [g_1 + \dots + g_k] \tag{5}$$

$$U_{V_d} = [h_1 + \dots + h_n] \tag{6}$$

Where:

$$g_j = g_j(u_1, u_2, \dots, u_m) \tag{7}$$

$$h_p = h_p(\dot{u}_1, \dots, \dot{u}_m) \tag{8}$$

Where U_{V_s} is a part of non-inertial energy that is stored and U_{V_d} is the remaining that is dissipated. Equation (4) has the main idea of equation (1) which describes the performed process based on energy components in viscoelastic bodies. In equation (5), the functions of g_j show the dependent components as a function of independent components while the functions of h_p show the effects of applying energy rate on the structure of non-inertial energy of body. In a viscoelastic body, functions of g_j are activated because of the poisson's ratio of material while functions of h_p are activated because of the viscosity properties.

The variation of equation (4):

$$\delta U_V = \sum_{i=1}^m \delta u_i + \sum_{j=1}^k \sum_{i=1}^m \frac{\partial g_j}{\partial u_i} \delta u_i + \sum_{p=1}^n \sum_{i=1}^m \frac{\partial h_p}{\partial \dot{u}_i} \delta \dot{u}_i \tag{9}$$

The eq. (9) means that an amount of energy variation δU_T is given to the body and the variation of energy components due to this process are considered, so as a result the following changes will be occur in the independent components :

$$\delta u_1 = \alpha_1 \delta U_T$$

$$\vdots$$

$$\delta u_m = \alpha_m \delta U_T \quad (10)$$

The coefficients α_i as loading coefficients will be depended on how energy is applied to the body as well as materials properties of the body, therefore:

$$\delta \dot{u}_i = \dot{\alpha}_i \delta U_T + \alpha_i \delta \dot{U}_T \quad (11)$$

By assuming that:

$$\alpha_s = \left[\sum_{i=1}^m \alpha_i \left(1 + \sum_{j=1}^k \frac{\partial g_j}{\partial u_i} \right) \right] \quad (12)$$

$$\alpha_d = \left[\sum_{i=1}^m \dot{\alpha}_i \left(\sum_{p=1}^n \frac{\partial h_p}{\partial \dot{u}_i} \right) \right] \quad (13)$$

$$\alpha = \alpha_s + \alpha_d \quad (14)$$

$$\beta = \sum_{i=1}^m \alpha_i \left(\sum_{p=1}^n \frac{\partial h_p}{\partial \dot{u}_i} \right) \quad (15)$$

The variation of non-inertial energy can be rewritten as follows:

$$\delta U_V = \alpha \delta U_T + \beta \delta \dot{U}_T \quad (16)$$

And also:

$$\delta U_{V_s} = \alpha_s \delta U_T \quad (17)$$

$$\delta U_{V_d} = \alpha_d \delta U_T + \beta \delta \dot{U}_T \quad (18)$$

By using equation (3), the variation of the residual energy in a performed process can be written as follows:

$$\delta U_R = \delta U_T - \delta U_V \quad (19)$$

By placing the equation (16) in (19):

$$\delta U_R = (1 - \alpha) \delta U_T - \beta \delta \dot{U}_T \quad (20)$$

Equation (20) gives the residual energy of the body as a function of the amount and rate of applied energy to the body as well as coefficients α and β .

3. LINEAR VISCOELASTICITY

In this part, using the presented approach, some basic linear viscoelasticity models are investigated and obtained results will be compared with the expected ones. At first, a viscoelastic body under uniform uniaxial stress, as shown in figure 1, is considered. This example can show the basis of presented approach clearly.



Figure 1: Viscoelastic body under uniform uniaxial stress

Because of uniaxial stress, Poisson's ratio does not have an effect on the non-inertial energy in this problem, so there is no need to use functions g_j . If we consider the effects of viscosity as a dependent component and also use a linear model, then the following statement can be used to express non-inertial energy:

$$U_V = u_e + c_1 \dot{u}_e + c_2 \quad (21)$$

Where u_e is the elastic energy as independent component and $h_e = c_1 \dot{u}_e$ is used to consider the effect of linear viscosity of body as dependent component. Also c_1 and c_2 are constant, where c_1 depends on the material's viscosity and elastic material properties and c_2 depends on the initial conditions.

By using of equations (14) and (15):

$$\alpha = \alpha_e + c_1 \dot{\alpha}_e \quad (22)$$

$$\beta = c_1 \alpha_e \quad (23)$$

Which α_e represents the proportion of energy that is converted to elastic energy. Therefore equation (16) can be rewritten as follows:

$$\delta U_V = (\alpha_e + c_1 \dot{\alpha}_e) \delta U_T + (c_1 \alpha_e) \delta \dot{U}_T \quad (24)$$

By defining how energy is applied to body, the variation of non-inertial energy can be calculated. The dissipated and stored parts of the non-inertial energy are also as follows:

$$\delta U_{V_s} = (\alpha_e)\delta U_T \quad (25)$$

$$\delta U_{V_d} = (c_1\dot{\alpha}_e)\delta U_T + (c_1\alpha_e)\delta \dot{U}_T \quad (26)$$

Equations (25) and (26) state that the stored part depends on the amount of applied energy only while the dissipated part depends on the amount as well as rate of the applied energy. This is in line with the definition of viscoelastic materials. Equation (26) also takes a clear description to the dissipated energy and its reasons. Figure 2 shows linear viscoelastic kelvin model.

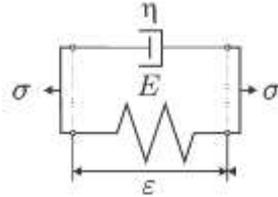


Figure 2: Kelvin-Voight model

For elements shown in figure 2 based on the parameters defined in equation (21):

$$\frac{\partial u_e}{\partial \varepsilon} = E\varepsilon \quad (27)$$

$$\frac{\partial u_d}{\partial \varepsilon} = \eta\dot{\varepsilon} \quad (28)$$

That u_e and u_d are stored and dissipated energies respectively, therefore:

$$\frac{\partial^2 u_e}{\partial \varepsilon \partial t} = \frac{E}{\eta} \frac{\partial u_d}{\partial \varepsilon} \quad (29)$$

$$u_d = \frac{\eta}{E} \dot{u}_e + f(t) \quad (30)$$

Where $f(t)$ is a function of time related to initial condition. To determine α_e as loading coefficient for viscoelastic kelvin model, it can be written based on energy conservation principle and equations (21) and (30):

$$\delta U_T = \delta u_e + \delta u_d \quad (31)$$

Where:

$$\delta u_d = c_1 \delta \dot{u}_e = \frac{\eta}{E} \delta \dot{u}_e \quad (32)$$

$$\delta u_d = \left(\frac{\eta}{E}\right) [\dot{\alpha}_e \delta U_T + \alpha_e \delta \dot{U}_T] \quad (33)$$

Therefore:

$$\delta U_T = (\alpha_e)\delta U_T + \left(\frac{\eta}{E}\right) [\dot{\alpha}_e \delta U_T + \alpha_e \delta \dot{U}_T] \quad (34)$$

$$\dot{U}_T = (\alpha_e)\dot{U}_T + \left(\frac{\eta}{E}\right) [\dot{\alpha}_e \dot{U}_T + \alpha_e \ddot{U}_T] \quad (35)$$

That yield:

$$\dot{\alpha}_e + \left[\frac{\dot{U}_T + \left(\frac{\eta}{E}\right)\ddot{U}_T}{\left(\frac{\eta}{E}\right)\dot{U}_T} \right] \alpha_e = \left(\frac{E}{\eta}\right) \quad (36)$$

Therefore, α_e can be calculated from equation (36) which strongly depends on how energy would be applied to the system.

As another way, if dissipated energy is considered as independent component:

$$U_V = u_d + c_1 \dot{u}_d + c_2 \quad (37)$$

Therefore:

$$\alpha = \alpha_d + c_1 \dot{\alpha}_d \quad (38)$$

$$\beta = c_1 \alpha_d \quad (39)$$

$$\delta U_V = (\alpha_d + c_1 \dot{\alpha}_d)\delta U_T + (c_1 \alpha_d)\delta \dot{U}_T \quad (40)$$

This results in:

$$\delta U_{V_s} = (\alpha_d)\delta U_T \tag{41}$$

$$\delta U_{V_d} = (c_1 \dot{\alpha}_d)\delta U_T + (c_1 \alpha_d)\delta \dot{U}_T \tag{42}$$

Equation (37) expresses the energy structure of Maxwell viscoelastic model, as shown in figure 3. For this model:

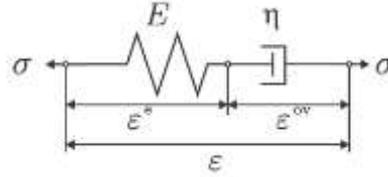


Figure 3: Viscoelastic Maxwell model

$$\frac{\partial u_d}{\partial \varepsilon^{OV}} = \eta \dot{\varepsilon}^{OV} = E \varepsilon^e = \frac{\partial u_e}{\partial \varepsilon^e} \tag{43}$$

By supposing that second derivative of ε^{OV} with respect to time is zero, this equation yields:

$$u_e = \frac{\eta}{2E} \dot{u}_d \tag{44}$$

Therefore, for this model, the loading coefficient α_d can be obtained as bellows:

$$\dot{\alpha}_d + \left[\frac{\dot{U}_T + \left(\frac{\eta}{2E}\right)\dot{U}_T}{\left(\frac{\eta}{2E}\right)\dot{U}_T} \right] \alpha_d = \left(\frac{2E}{\eta}\right) \tag{45}$$

As seen the general form of non-inertial energy structure of Kelvin and Maxwell models are similar.

This procedure could be generalized for an arbitrary kelvin elements connected in series as shown in figure 4. The non-inertial energy structure is:

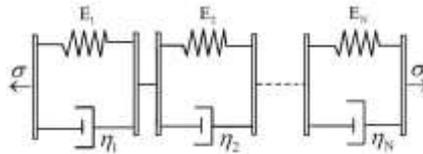


Figure 4: Generalized kelvin elements

$$U_V = u_{e1} + u_{e2} + \dots + u_{eN} + c_{e1}\dot{u}_{e1} + c_{e2}\dot{u}_{e2} + \dots + c_{eN}\dot{u}_{eN} + constant \tag{46}$$

This is also true for generalized Maxwell elements, as shown in figure 5:

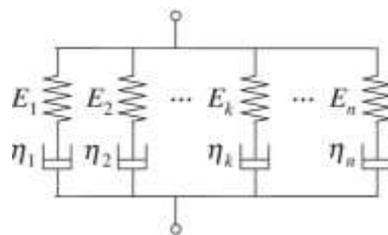


Figure 5: Generalized Maxwell element

$$U_V = u_{d1} + u_{d2} + \dots + u_{dN} + c_{d1}\dot{u}_{d1} + c_{d2}\dot{u}_{d2} + \dots + c_{dN}\dot{u}_{dN} + constant \tag{47}$$

Indeed obtained non-inertial energy equations show that when an amount of energy is applied to the system, what would be the feasible changes of the system based on the way the energy is applied to it.

4. CONCLUSIONS

Based on the way considering independent and dependent energy components, an energy distribution to the system could be presented to consider the physical properties as well as feasible directions.

The whole energy of the body is divided into two parts: residual energy and non-inertial energy. The non-inertial energy equation depends on the physical properties of the body and its variation is defined based on the amount and rate of applied energy to the system as well as the loading coefficients.

By considering energy conservation principle, the non-inertial energy as well as residual energy could be calculated. The presented approach applied to linear viscoelastic models which resulted in obtaining a first order differential equation to calculate dissipated or storing energy based on the way that the whole energy is applied to the system, in other words energy transfer rate and acceleration. For Kelvin and Maxwell models, similar general form of non-inertial energy equation was obtained.

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