Students' Ability and Achievement in Recognizing Multiple Representations in Algebra

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ABSTRACT--- ***Secondary school students often demonstrate a degree of proficiency manipulating algebraic symbols, when learning linear relationships with one unknown. They, in some cases and when encouraged verbalize and explain the steps taken, thereby demonstrating awareness of the procedures with symbols according to fixed rules. It is well known that correct procedural skills are not always supported by conceptual understanding. Previous research suggests that one of the indicators of conceptual understanding is the ability to structurally recognize the same relationship posed through multiple representations. The purpose of this study is to examine the relationship between secondary school students' achievement on standardized test and their ability to recognize structurally the same relationship presented in different forms and their ability to solve problems involving linear relationships with one unknown presented in different ways. The study was conducted with a large sample size (N=300), of senior secondary school class two(SS2) students from Bwari Area Council of Federal Capital Territory, Abuja, Nigeria, using questions drawn from past Educational Resource Center's (ERC) past promotion examination questions. It was observed that there were positive but weak correlations (ρ=0.114) between, the students' examination scores and their ability to identify the same relationship posed in different modalities, the students' examination scores and their ability to solving problems in all problem sets, the students' solved problems posed in different modalities and their ability to identify the same relationship posed in different modalities, using both Pearson and Spearman's correlations. It is recommended that teachers should emphasize multiple-representation in algebra in the classes they teach.***

Keywords--- ability, recognize, multiple-representation, achievement, algebra

1. INTRODUCTION

In learning about linear relationships with one unknown, secondary school students often demonstrate some degree of competency in manipulating algebraic symbols. When encouraged, they can verbalize and explain the steps taken, thus demonstrating awareness of procedures with symbols according to fixed rules. Students show a certain level of operational conception (Sfard, 1991) or process conception, (Dubinsky and McDonald, 1991), however they do not necessarily reveal structural conception (Sfard, 1991) or object conception, (Dubinsky and McDonald, 1991). Correct and seemingly fluent procedural skills are not always substantiated by conceptual understanding (Herscovics, 1996; Langrall and Swafford, 1997). When the students demonstrate the ability to recognize structurally the same relationship/concept presented in different modalities (verbal, diagrammatic and symbolic), it is likely they have developed conceptual understanding of the relationship/concept and advanced from procedural skills (process conception) to structural skills (object conception), (Panasuk, 2010). Having conceptual understanding enables students to meaningfully operate upon rules and procedures, and provides a strong basis for effective problem solving. Superior representational knowledge is likely to be associated with higher performance on complex tasks requiring principled understanding of mathematical concepts (Neimi, 1996).

1.1 Algebraic reasoning and conceptual understanding in algebra

The term algebraic reasoning is used to describe mathematical processes of generalizing a pattern and modeling problems with various representations (Herbert and Brown, 1997; Driscoll, 1999). Another definition of algebraic reasoning is the capacity to represent quantitative situations so that relations among variables become apparent (Driscoll,1999). Algebraic reasoning is the ability to operate on an unknown quantity as if the quantity is known (Swafford and Langrall, 2000). Algebraic reasoning is a way of reasoning involving variables, generalizations, different modes of representation, and abstracting from computations, (Vance, 1998). These definitions provide the foundation for drawing conclusions and explanations about conceptual understanding in algebra in this work. Conceptual understanding in algebra is the ability to recognize functional relationships between known, and unknown, independent and dependent variables, and to distinguish between and interpret different representations of the algebraic concepts. Competency in reading, writing, and manipulating both number symbols and algebraic symbols are the manifestations (Panasuk and

Beyranevand, 2010). Those who show conceptual understanding grasp the full meaning of the concept, and can discern, interpret, compare and contrast related ideas of the subtle distinctions among a variety of situations. Fluency in the language of algebra demonstrated by confident use of its vocabulary and meanings and flexible operation upon mathematical properties and conventions indicate conceptual understanding in algebra.

1.2 Representations

Representations and symbol systems are fundamental to mathematics as a discipline. Mathematics is inherently representational in its intentions and methods (Kaput, 1989).

The process of representation involves identification, selection and presenting one idea through something else (Seeger, 1998). Presentation through pictures symbols and signs (notations) that are structurally equivalent might be referred to as representation. An attribute of mathematical concepts, which are defined by three variables: the situation that makes the concept useful and meaningful, the operation that can be used to deal with the situation and the set of symbolic, linguistic, and graphic representation that can be used to represent situations and procedures is another way of viewing representation (Vergnaud, 1997).

Bruner (1966) proposed to distinguish three different modes of mental representation – the sensory-motor (physical action upon objects), the iconic (creating mental images) and the symbolic (mathematical language and symbols). Mathematical representations are internal abstraction of mathematical ideas or cognitive schemata (Pape and Tchoshanov, 2001). The learner constructs to establish internal mental network or representational system, (Hiebert and Carpenter, 1992).

The notion of multiple representations in mathematics education has evolved considerably in recent years, and different theories of representations utilize different terminology, (Goldin and Shteingold, 2001). Associated with representations in mathematical language are ordinary language, mathematical verbal language, symbolic language, visual representation, unspoken but shared assumptions and quasi-mathematical language (Pirie, 1998). She asserted that the function of any type of representation is to communicate mathematical ideas, and that each representational system adds to effective communication and helps to convey different meanings of a single mathematical concept.

The research in the area of representation has been focused on student generated representations and subsequent impact of these representations on learning mathematical concepts (Ainsworth, et. al., 2002; Diezmann, 1999; Diezmann and English, 2001; Lowrie, 2001; Swafford and Langrall, 2000). When students generate representations of a concept or while solving problems (as a means of mathematical communication) they naturally tend to reduce the level of abstraction (given by the problem) to a level that is compatible with their existing cognitive structure, Pape and Tochanov (2001).

Children first learn about an object by acting upon it and through interaction they eventually understand its nature (Piaget, 1970). They distinguish between a process conception or operational conception and an object conception or structural conception of mathematical principles and notions, and agree that when a mathematical concept is learned, its conception as a process precedes its conception as an object. These theories also suggest that the process conception (e.g. simply following or performing the steps shown by teacher) is less abstract than an object conception (the nature of the concept with its properties, rules and understanding of how and why the rules work), (Sfard, 1992; Dubinsky and McDonald, 1991). Thus, the process conception of a mathematical concept can be interpreted as being on a reduced level of abstraction than its conception as an object. When students need to reduce the level of abstraction, it is likely that they have not yet developed conceptual understanding.

1.3 Purpose of the study

The purposes of this study were to:

1. Investigate the relationship between secondary school students' level of achievement and their ability to recognize and solve problems that are structurally the same.
2. Investigate the students' understanding of linear relationship with one unknown posed in different ways (symbols, diagrams and words).

1.4 Significance of the study

The current study is important because the researcher is not yet aware of known studies on students' ability and achievement in recognizing multiple representations in algebra concerning Government Secondary School Bwari, Federal Capital Territory Abuja, Nigeria. It will help in improving the teaching and learning of algebraic concepts and their application in real life and in the communities surrounding Bwari Area Council of Federal Capital Territory, Abuja, Nigeria.

1.5 Research Questions

The following questions are stated to guide the study:

1. Is there a correlation between the students' Exam. Scores and their ability to identify the same relationship posed in different modalities, using Spearman's correlation?
2. Is there a correlation between the students' Exam. Scores and their ability to identify the same relationship posed in different modalities, using Pearson's correlation?
3. Is there a correlation between the students' Exam. Scores and their ability to Solving problems in all problem Sets, using Spearman's correlation?
4. Is there a correlation between the students' Exam. Scores and their ability to Solving problems in all problem Sets, using Pearson's correlation?
5. Is there a correlation between the students' Solved problems posed in different modalities and their ability to identify the same relationship posed in different modalities, using Spearman's correlation?
6. Is there a correlation between the students' Solved problems posed in different modalities and their ability to identify the same relationship posed in different modalities, using Pearson's correlation?

2. METHODOLOGY

The method use in this work is non-experimental mixed methodology design involving a multi-component survey and interviews with selected students.

2.1 Sample

Three hundred (*N* =300) senior secondary school students from Bwari Area Council of Federal Capital Territory, Abuja, Nigeria, took part in the survey. Students from this area council have been taking WAEC examinations with majority coming out with poor grades, hence this study. Educational Resource Center's (ERC),Abuja Nigeria, reports examination scores in nine (9) categories, which include, Fail (F9:0-39), Pass (E8: 40-44 and D7: 45-49), Credit (C6:50-54; C5:55-59; C4:60-64), Good (B3:65-69), Very Good (B2:70-74) and Excellent (A1:75-above), like the West African School Certificate Examination Council (WAEC).

Table 1:Participating Students' Exam Scores

Tier	Exam Scores Classification	Frequency	Percentage
Fail	F9: 0-39	40	13.33
Pass	E8: 40-44	55	18.33
	D7: 45-49	60	20.00
Credit	C6:50-54	48	16.00
	C5:55-59	34	11.33
	C4:60-64	28	9.33
Good	B3:65-69	15	5.00
V. Good	B2:70-74	12	4.00
Excellent	A1:75-above	8	2.67

2.2 Subjects

The study was conducted with a sample size (*N* =300) of senior secondary school class two students (SS2) from Bwari Area Council of Federal Capital Territory, Abuja, Nigeria. This simple was drawn by using their past examination results and questions drawn from the Educational Resource Center's (ERC) past promotion examination questions to test the students.

The role of multiple representations, in probing understanding of mathematics learning, development of algebraic reasoning and solving problems posed in different representational ways are of great importance in learning Algebra, hence forming the foundation of this study.

The compositions of the samples are shown in table 1 above.

2.3 Instrument

The original instrument, designed by the researchers (Panasuk, 2006, 2010; Beyranevand, 2010) for a multiyear large scale study consisted of several interrelated parts. This paper reports on the data collected from two parts; Part R (Recognition) and Part P (Problems). These parts of the instrument measured both students' ability to recognize structurally the same linear relationships and to solve problems that involve linear relationships with one unknown posed through multiple representations. The data collected from both parts were correlated with the students' achievement level based on their scores graded according to the ERC scores ratings. Educational Resource Center's (ERC) past promotion examination questions were used because commonly used achievement tests provide at best only indirect and highly limited information on students' understanding of specific conceptual domains,(Neimi,1996).

Since conceptual understanding in Algebra was one of the foci of this study, testing and measuring both the ability to recognize and to solve problems involving structurally the same linear relationship with one unknown posed in different modalities strengthened the analysis of the data and validated the findings.

Figure 1 illustrates Recognition, which involved structurally the same linear relationship with one unknown posed in three different representations.

S.	**D.**	**W.**
$x + 25 = 52$	52 •…………•……………….• 25 x	Silas is 25 years older than his son. He is now 52years old. How old is his son now?

Figure 1: Recognition instrument

The students were asked to observe (not solve the problems) and explain in writing if they recognize some similarities between the problems presented in three different modes. Ideally, they were expected to state that each problem represents structurally the same linear relationship, i.e., the sum of 25 and an unknown quantity is 52.

Another three sets, each of which consisted of three problems (total nine problems): Problem Set S (symbols), Problem Set D (diagrams), and Problem Set W (words), similar to the problems shown in Figure 1. The first two problems in each Problem Set could be solved in one-two steps; the third problem in each set required more than two steps. The problems in each set have their counterparts in other sets presented in different ways. The students were asked to solve each problem. For each Problem Set a coding system was created.

Followed by the analysis of the data, fifteen most representative surveys were selected and the students were invited for interviews. The selection was based on the responses that had potential to provide information related to the different achievement levels and the students' ability to recognize structurally the same linear relationship and solve problems. Eight of the fifteen students agreed to be interviewed.

3. RESULTS ANALYSES AND DISCUSSIONS

All collected surveys were coded on a nominal or ordinal scale, frequencies, means, and standard deviations were calculated for each part of the instrument; regression analysis was completed to quantify the relationship between the variables.

The scoring frequencies of the students' responses were measured as ordinal data, grouped in four categories and compared with the corresponding ERC's promotion examination score for each student (Table 2). Students who did not recognize that three problems in the Recognition test represented structurally the same relationship (answered 'No') or left the space blank, were grouped together into category 1.

The next group of students (category 2) indicated neither 'Yes' nor 'No' but used vague unspecific language attempting to provide some description that could have been considered relevant. The category 3 included students who answered 'Yes' and made an attempt to describe similar features of the problems. They either indicated that the problems had the same numbers, or described the operation needed to solve the problems, or recognized that all the problems had the same numerical solution. The category 4 was formed with students who clearly identified and explained that all three problems represent structurally the same relationship.

It must be commented here that the responses that fell into the categories 1 and 4 were relatively easy to distinguish and classified. However, scoring of the responses in the categories 2 and 3 was particularly difficult due to students' vague and unspecific language. To ensure certain level of scoring trustworthiness, each response was re-examined for consistency. What actually mattered was whether the students did not recognized (category 1), distinctively recognized and explained (category 4), and/or produced some explanations, yet unambiguous (category 3 and 4). Thus the categories 3 and 4 were combined together for the purpose of analysis.

Table 2: Exam categories and Recognition frequency

Coding Groups	SCORES-FREQUENCY/PERCENTAGE																	
	Fail		Pass				Credit						Good		V. Good		Excellent	
	F9		E8		D7		C6		C5		C4		B3		B2		A1	
	N	%	N	%	N	%	N	%	N	%	N	%	N	%	N	%	N	%
Group4: Recognized the same relationship &explained	**0**	**0**	**0**	**0**	**3**	**7.7**	**5**	**13.7**	**3**	**7.7**	**2**	**5.3**	**6**	**15.9**	**12**	**31.8**	**8**	**21.2**
Group 3: Recognized similarities, but not explicitly identified the same relationship	**5**	**4.1**	**20**	**16.4**	**25**	**20.5**	**25**	**20.5**	**20**	**16.4**	**16**	**13.1**	**6**	**4.9**	**8**	**6.6**	**2**	**1.6**
Group 2: Used vague unspecific language	**15**	**13.2**	**28**	**24.6**	**30**	**26.3**	**18**	**15.8**	**11**	**9.6**	**6**	**5.3**	**3**	**2.6**	**2**	**1.8**	**1**	**0.9**
Group 1: Did not recognize the same relationship	**20**	**60.6**	**7**	**21.2**	**2**	**6.1**	**0**	**0**	**0**	**0**	**4**	**12.1**	**0**	**0**	**0**	**0**	**0**	**0**

Almost all (except four) students in the category 1 were from the fail and pass tier, and twenty-six of the good, very good and excellent tiers students were in the category 4. The majority of the category 2 and 3 students were from the credit tier. About 13.33% of all participating students formed the category 4, which indicates that not all students in the credit, good, very good and excellent tiers were able to recognize structurally the same relationship presented in different modalities, thus showing that some students still have problems in their conceptual understanding of the concept.

Pearson and Spearman correlations between the ability to recognize and achievement level yielded a weak, yet statistically significant positive correlation (Table 3).

Table 3: Recognition of the same Linear Relationship: Pearson and Spearman Correlations

Dependent Variable	Independent Variable	Spearman's	Sig.	Spearman's	Sig.
Identified the same relationship posed in different modalities	Exam. Scores	0.142	0.005*	0.150	0.003 *

Note: * indicates a significance value < 0.05.

Table 4 shows the mean, standard deviation and frequencies of the students' correct solutions for each problem. Each student's score is based on the number of responses answered correctly for each of the three problems in each Problem Set.

Table 4: Recognition: Problem Set Frequencies

Problem Set	Problem correct			Mean of all three problems	Standard deviation
	Q 1	Q 2	Q 3		
S:Symbolic representation	85.9%	82.7%	80.7%	2.62	0.731
D:Diagrammatic representation	80.0%	78.7%	65.6%	2.36	0.724
W: Verbal representation	76.7%	78.2%	49.2%	2.34	0.634

The data revealed that students were most successful solving the equations represented symbolically and less successful finding the unknown length of the line segment.

To compare the students' test scores with their ability to solve problems posed in three different modalities, Pearson and Spearman correlations were calculated (see Table 5).

Table 5: Pearson and Spearman Correlation for solving all problems in the Problem Sets

Dependent Variable	Independent Variable	Spearman's	Sig.	Pearson	Sig.
Solving problems in all problem Sets	Exam. Scores	0.487	0.000*	0.481	0.000 *

Note: * indicates a significance value < 0.05.

The results yielded a strong positive correlation with a significance of $p < 0.001$. The students who were able to correctly solve problems in posed in three different representations were significantly more likely to be in the credit, good, very good and excellent tiers.

To understand the association between the students' ability to recognize structurally the same relationship and solve problems in each of the same representations, Pearson and Spearman were calculated (see Table 6).

Table 6: Correlations for Recognition of the Same Relationship and Solving Equations

Dependent Variable	Independent Variable	Spearman's	Sig.	Pearson	Sig.
Solved problems posed in different modalities	Recognized the same relationship posed in different modalities	0.114	0.031*	0.126	0.011 *

Note: * indicates a significance value < 0.05.

While the correlation is weak, it is statistically significant to conclude that the students who recognized structurally the same relationship were likely to be able to solve problems involving linear relationship with one unknown posed in different modalities.

In summary, the results showed that there is an association between students" achievement based on the standardized tests and their ability to recognize and solve problems involving structurally the same linear relationship with one unknown presented in different modes (symbols, diagrams, and words).

3.1 Qualitative Component

A semi-structured format interviews with nine selected students provided another layer of evidence. How the students solved the problems posed in multiple representations informed our understanding of the students' ability to recognize multiple representations of structurally the same concept, translate between the representations and communication of their reasoning in a perceptible mathematics language.

First, the students were encouraged to look at their Part R's responses and verbalize the relationship in each of the three problems (Figure 1). Then, the students we encouraged to create diagrams for the problem #2 from both Set W (words) and Set S (symbols), and write algebraic equations for same problem from both the Set W (words) and Set D (diagram).

3.1.1 Fail, Pass and Credit groups.

The fail, pass and credit tiers students used trial and error method to solve problems from the Set W and Set S. It is likely that these students have not transitioned from iconic to the symbolic mode of thinking, (Bruner, 1966). When encouraged to draw a diagram that would represent linear relationship described in words, they produced the drawings that showed rather the solution process to assist their calculations than algebraic structure of the relationship.

These students had difficulties in abstracting from computations and showed a persistent tendency to reduce the level of symbolic abstraction to the level of numerical (lower level) abstraction,(Pape and Tochanov, 2001). When solving algebraic equations. Their presentations showed they are still thinking in arithmetic terms by undoing the chain of operations mainly showing calculation in a column format or in the form of the numerical equation. Since these students had not developed reasoning abilities that were necessary for structural comprehension of linear relationship with one unknown, they were not able to recognize the structure of the relationship between quantities at a more abstract level than numerical.

Some of the students in this tier were able to find the correct solutions to the equations presented in symbols and words by using instantiations and/or manipulating with numerical and/or algebraic symbols (Problem Set W and S), however it was quite obvious that they were having difficulty or were unable to operate upon unknown quantity presented in a diagrammatic mode (Problem Set D). This explains why only few could recognize the relationships presented via diagram in the problem Set D. As a result, the lack of the diagrammatic skill (Diezmann, 1999, 2002) created a barrier for their meaningful learning (Kieran, 1992) and development of conceptual understanding.

3.1.2 Credit, Good, V. Good and Excellent groups

The major characteristic of the presentations of the students in these tiers was the ability to fluently transit between representations. These students solved all nine Part P problems correctly using algebraic methods and showed flexible use of each mode of representation. They able to make connections between the representations logically interpreted and translated among representations.

These students operated at the higher level of reflective abstraction, i.e., structural awareness, (Cifarelli, 1988). They had blended the process conception and the object conception by showing the ability to think and act upon the problems" structures (Sfard, 1991, 1992; Tall, 2008).

However, not all the students were able to recognize structurally the same relationship with one unknown in the Part R (Recognition). It was noticeable that these students were lacking solid understanding of the deep structure of the linear relationship, the nature of unknown, and presumably mechanically used the rules and followed the steps.

4. FINDINGS

1. The study revealed that students were most successful solving the equations represented symbolically and less successful finding the unknown length of the line segment and wobbled in word problems.

2.The results yielded a strong positive correlation with a significance of $p < 0.001$, using both Pearson's and Spearman's correlation coefficients between the dependent variable (Solving problems in all problem sets) and independent variable (Exam. scores).

3. There was a weak but positive correlation with significance of $p < 0.001$, using both Pearson's and Spearman's correlation coefficients between the dependent variable (Solved problems posed in different modalities) and independent variable (Recognized the same relationship posed in different modalities).

5. CONCLUSION

In this study, it was revealed that the students who were able to show conceptual understanding were most likely to perform at a higher level on the ERC's standardized test.

These students were able to interpret, connect and translate with confidence among the representations of the same relationship, which in turn is likely to lead to higher achievement, as suggested by Panasuk (2010). It was found that the achievement level is not a strong indicator of the conceptual understanding. Some of the students who were able to correctly solve linear equations with one unknown did not necessarily show the ability to recognize the same relationship presented in symbols, as a diagram, and/or in words. The results of the interviews showed that the students could manipulate symbols but revealed little conceptual understanding. It could be that these students have little exposure to diagrammatic training, in particular, and multiple representations, in general. Therefore, the findings in this work support previous researches which emphasized the benefits of encouraging students to use multiple representations when learning concepts and solving problems,(Moseley, 2005; Niemi, 1996).

In a called for diagrammatic literacy it was suggested that students must be exposed to systematic instruction which would help them to move beyond the surface or artistic representation of information to a more structural and sophisticated representation of the problem information (Diezmann and English, 2001).

Pictures provide an external sketchpad where students can represent and connect pieces of information. Generating pictorial representations facilitate the conceptualization of the problem structure and form the basis for a solution (Van Essen and Hamaker,1990). Pictures facilitate the reorganization of information and help in making implicit problem information explicit (Larkin and Simon,1987).

This study supports the assertion that the more diverse the students' representational knowledge is, the more likely they are able to produce correct solutions to problems and that exposure to numerous representations assists students in developing their mathematical knowledge.

6. RECOMMENDATION

From the findings stated above, it is recommended that;
1. Students be taken through an organized learning process from systematic and consistent learning to how to recognize the same relationship and solve problems posed in different representational modes.
2. The information to be learned in the classroom must be consistently and explicitly presented to the students in multiple ways in order for them to be able to build up a variety of thinking methods and techniques as these will enhance their cognitive structures.

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