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GENERALIZED P-KRULL DOMAINS

WAHEED AHMAD KHAN

Abstract. In [7] author (with A. Taouti) introduced P-Krull monoids and in [8] we presented P-Krull domains. As a result of bridging link the above, we came up with generalized P-Krull domains. However, generalized P-Krull domains are weak but simultaneously parallel to generalized Krull domains.

An integral domain R is said to be a pseudo-valuation domain (PVD) if every prime ideal of R is a strongly prime [1, Definition, page 2]. Let R be an integral domain having quotient field K. A prime ideal P of R is called strongly prime if $xy \in P$, where x, $y \in K$, then $x \in P$ or $y \in P$. An integral domain R is a PVD if and only if for each nonzero $x \in K \backslash R$ and for each nonunit $a \in R$, we have $ax^{-1} \in R$ [1, Theorem 1.5(3)]. Every valuation domain is a PVD [1, Proposition 1.1] but converse is not true. A quasi-local domain (R, M) is a PVD if and only if $x^{-1}M \subset M$ whenever $x \in K \backslash R$ [1, Theorem 1.4]. A Noetherian domain R with quotient field K is a PVD if and only if $x^{-1} \in R^{'}$(integral closure of R in K) whenever $x \in K \backslash R$ [1, Theorem 3.1].

Let R be an integrally closed domain with quotient field K and $F = \{v_\lambda\}_{\lambda \in \gamma}$ be a family of valuation overrings of R. Then R is said to be a Krull domain if, $R = \cap_\lambda V_\lambda$, each V_λ is DVR, the family F has finite character (that is, if $0 \neq x \in K$, then x is a non-unit in only finitely many of the valuation rings in the family F) and each V_λ is essential for R (A valuation over ring of integral domain R is said to be essential for R if V is fraction ring of R).

Here we recall some well known and basic definitions from the literature. An integral domain R with quotient field K is said to be root closed if whenever $x^n \in R$ for some $x \in K$, and $n \in \mathbb{Z}^+$, then $x \in R$. We define R to be n-root closed if whenever $x^n \in R$ for some $x \in K$, then $x \in R$. An integral domain R is root closed if and if only if it is n-root closed for each $n \in \mathbb{Z}^+$. More generally, if $R \subseteq B$ be a unitary commutative ring extension, then R is n-root closed in B if whenever $x \in B$ with $x^n \in R$, then $x \in R$. If B is the total quotient ring of R, we say R is $n - root\ closed$. A domain R is said to be completely integrally closed (in its quotient field K) if $cu^n \in R$ for all integers $n \geq 1$, $0 \neq c \in R$, $u \in K$ implies $u \in R$.

In [4] it has been proved that a Krull monoid domain $R[S]$ is an HFD if S has trivial class group while $CI(R) \cong \mathbb{Z}_2$. Further in [5] we (authors) have discussed the factorization properties, particularly the half-factorial and bounded factorization of Krull monoid domain $R[S]$ through the length functions and class groups of monoid S and domain R respectively.

We give introduction of generalized form of valuation map (i.e. pseudo-valuation map) established in [7], and presented the properties of a generalized P-Krull domains.

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We focused our attention on: If R is a Noetherian pseudo-valuation domain, then every overring of R is a pseudo-valuation domain [1, Corollary 3.3]. We considered R to be a root closed pseudo-valuation domain (corresponding to the pseudo-valuation map ω) and $F = \{V_\lambda\}_{\lambda\in\Lambda}$, the family of pseudo-valuation overrings of R, while constructing a Generalized P-Krull domain.

1. Generalized P-krull domains

In [8] we introduced pseudo-valuation map for P-Krull domains, here we generalize that map.

Definition 1. Let $\omega : K^* \to G$ be an onto map, which has the following properties. For $x,\, y \in K^*$;
(a) $\omega(xy) = \omega(x) + \omega(y)$.
(b) $\omega(x) < \omega(y)$ implies $\omega(x + y) = \omega(x)$.
(c) $\omega(x) = g \geq 0$ or $\omega(x) < \omega(y) = h$, where $g,\, h \in G$ and $h > 0$.

In Definition 1, the map ω is the extension of semi-valuation map. No doubt (b) implies that it is a quasi-local domain as discussed in [2, page 180]. Moreover condition (c) plays an important role and it induce property $*$ in G. Hereafter we call ω, the pseudo-valuation map.

Example 1. Every valuation map from quotient field of a ring R i.e., K to $K * /U(R)$ is clearly an example of pseudo-valuation map.

We know that every pseudo-valuation domain is necessarly be a local ring, also a locally pseudo-valuation (resp, locally divided) domain is a domain D such that, for each maximal ideal M of D, D_M is a pseudo-valuation (resp, divided) domain. A ring extension $R \subset T$ satisfies going-down if, whenever $p,\, q \in Spec(R)$ and $q_1 \in Spec(T)$ satisfy $p \subset q$ and $q_1 \cap R = q$, there is $p_1 \in Spec(T)$ such that $p_1 \cap R = p$ and $p_1 \subset q_1$. A going-down domain is a domain D such that, for each overring T of D, the extension $D \subset T$ satisfies going-down.We know about the importance of dimension of pseudo-valuation domains. We have to combine all such characteristics to develope a single domain.

Let us suppose that R be a root closed pseudo-valuation domain, and $F = \{V_\lambda\}_{\lambda\in\Lambda}$ be the family of pseudo-valuation overrings of R such that each V_λ containing R, satisfying going down and is root closed. Then we can easily write it as $R = \cap\, V_\lambda$ for further look forward, here we define few terminology.

Remark 1. Let P be a strongly prime ideal in D and G^* be a group of divisibility of D, then there is one to one corresponding between strongly prime ideals and a subsets X in G^* which generate the convex subgroups $B(G)$ as in [6, Remark 1]. By the definition of strongly prime ideal i.e P is strongly prime if and only if $x^{-1}P \subset P$ whenever $x \in K\backslash R$ we have convex set C, $x \in G\backslash C$ such that $-x+C \subset C$ and also generating a convex subgroup. We call such C a strongly convex set [6, Remark 2].
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Definition 2. (a) A family Ω of pseudo valuations of the field K is said to be of finite character if for every $x \in K, x \neq 0$, the set $\{\omega \in \Omega : w(x) \neq 0\}$ is finite.
(b) If a family Ω of pseudo valuations of the field K, $\omega \in \Omega$ has ring R_ω and a maximal ideal (strongly prime) P then the ring $A = \cap_{\omega\in\Omega} R_\omega$ is said to be defined

by Ω and $P \cap A$ is a (stongly) prime ideal called centre of ω on A, denoted by $Z(\omega)$. If $R_\omega = A_{Z(\omega)}$ then ω is said to essential pseudo valuation for A.

(c) Rank of pseudo-valuation domain D is the rank of pseudo-value group of D i.e G^*. Rank of pseudo-value group of G^* depend upon the existence of ordinal type of set of proper strongly convex sets which are described in [6, Remark 2] under inclusion ($\subseteq$).

Example 2. Let R be a noetherian pseudo-valuation domain, then each overing of R is pseudo-valuation ring. As each valuation ring is a pseudo-valuation ring, so by taking overring of R, and there is a valuation map, which satisfies each characteristics (a), (b),and (c)of definition*2.*

Remark 2. Let R be a root closed pseudo-valuation domain and $F = \{V_\lambda\}_{\lambda\in\Lambda}$ be the family of pseudo-valuation overrings of R such that each V_λ is n-root closed and contains R then, $R = \cap V_\lambda$.

After writing R as an intersection of family of pseudo-valuation overring we define the rank, dimension and character of ring R and pseudo-valuation overring of R we define.

Definition 3. Let R be a root closed pseudo-valuation domain, and $F = \{V_\lambda\}_{\lambda\in\Lambda}$ be the family of pseudo-valuation overrings of R such that each V_λ is n-root closed and contains R then,

(1) $R = \cap\, V_\lambda$.

(2) Each V_λ is of finite character (every non-zero element is contain in at most finitely many maximal ideals of of V_λ).

(3) Each V_λ has rank one.

(4) each V_λ is essential for R (PVD is essential for R if it is fraction ring of R).

In the above definition each characteristic can easily be prove by definition1. A domain R which satisfy the above conditions is called generalized pseudo-Krull domains (generalized P-Krull domains). If we look upon generalized P-Krull domains then we see that it is a pullback over V_λ/M where M is unique common maximal ideal in each overring of domain R. Thus we can write

$$\begin{array}{ccc} R = \phi^{-1}(D) & \rightarrow & D \\ \downarrow & & \downarrow \\ V_\lambda & \rightarrow & V_\lambda/M \end{array}$$

Where D is root closed subring of V_λ/M, then $R = D + M$ is generalized P-Krull domain.

Below is an example when generalized P-Krull domain is a PVD and also an atomic.

Example 3. Let B be a one-dimensional valuation domain of the form $K + M$, where K is a field contained in B and M is the maximal ideal of B. Also $k_1 \subset k_2 \subset \ldots \subset K$, where each k_i is root closed in K. Let k_1 be a root closed subfield of K, then the subring $A = k_1 + M$, clearly A is a root closed PVD and can be written as an intersection of $k_i + M$ where $i > 1$. Hence A is a generalized P-Krull domain (which is also an atomic PVD).

Remark 3. If V is a cononical valuation assosiated with pseudo-valuation and $V = K + M$, suppose D is root closed in V then $R = D + M$ is a generalized pseudo-valuation domain (generalized P-Krull domain) as follows.

$$\begin{array}{ccc} R = \phi^{-1}(D) & \rightarrow & D \\ \downarrow & & \downarrow \\ V & \rightarrow & V/M \end{array}$$

Since each valuation overring is a pseudo-valuation overring but converse is not true hence characteristics (2), (3) and (4) of definition3 are weaker than as for the Krull domain.

Remark 4. If each V_λ containing the finite number of prime ideals then rank of V_λ equal to the dimension. Then R is root closed one dimensional pseudo-valuation (local) domain, here maximal ideal should be strongly prime ideal.

Remark 5. If we assume that every overring of R is pseudo-valuation domain then we have added the noetherian property and R will be root closed one dimensional noetherian pseudo-valuation (local) domain.

Above remarks shows that we have closely related domains to generalized P-krull domains are:

(1) One dimensional noetherian domain is **weakly krull domain.**

(2) One dimensional noetherian local domain is DVR.

We have,

$$\begin{aligned} valuation\ domain &\Rightarrow pseudo-valuation\ domain \\ valuation\ map &\Rightarrow pseudo-valuation\ map \end{aligned}$$

thus we have: $Generalized\ krull\ domain\ (having\ every\ ideal\ is\ strongly\ prime) \Rightarrow Generalized\ P-krull\ domain.$

Here we provide a number of examples which are generalized P-Krull domains.

Example 4. (i) Let L be algebraic closure of Q and let k be a subfield of L consisting of all elements $\beta \in L$ such that minimal polynomial for β over Q is solvable by radicals over Q, let $\alpha \in L$ not in k and let $F = k(\alpha)$. Then $A = k + XF[[X]]$ is root closed one dimensional local noetherian domain and hence generalized P-Krull domain.

(ii) Let R be a quasi-local domain with maximal ideal M which satisfies $x^{-1}M \subset M$. Let S_α be set of over rings of R each is quasi-local having common maximal ideal M which is storngly prime and each containing R. Then by clearly R/M is root closed in each S_α/M, and is clearly a generalized P-krull domain*3.*

(iii) If B be a one-dimensional valuation domain of the form $K + M$, where K is a field contained in B and M is the maximal ideal of B. Also $k_1 \subset k_2 \subset ... \subset K$ where each k_i is root closed in K. Let k_1 be a root closed subfield of K, then the subring $A = k_1 + M$, then clearly A is a root closed pseudo valuation domain and can be written as an intersection of $k_i + M$ where $i > 1$ This A is clearly generalized P-krull domain.

(iv) Also if R is generalized P-Krull domain and T be domain extension such that $spec(R) = spec(T)$, if R is n-root closed in T then R/P is generalized P-Krull in T/P.

Basically the above generalized P-Krull domain are arising in the case when each overring is pseudo-valuation. We have a lot of literature which reflects that when an overrings of pseudo-valuation domains are pseudo-valuations? see in [3, Propositions 3.4, 3.5, 3.14.].

Proposition 1. *Generalized Krull domain having every ideal a strongly prime, is a generalized P-Krull domain.*

Proof. Let R be a generalized Krull domain no doubt it is integerally closed hence root closed, also each generalized Krull domain can be written as an intersection of valuation overrings. As valuation overrings implies pseudo-valuation overrings thus (1) of definition3 is satisfied similarly easy to prove (2), (3) and (4) of definition3 which consitute the proof. □

Proposition 2. *Let R is generalized P-Krull domain and $F = \{V_\lambda\}_{\lambda\in\Lambda}$ family of pseudo-valuation overrings as in (definition3 (1)). if each V_λ containing maximal ideal M_α then $\cap M_\alpha$ is non-empty atleast $\cap M_\alpha = c$ such that $c^{-1} \in F = \{V_\lambda\}_{\lambda\in\Lambda}$ for each $\lambda \in \Lambda$.*

Proof. Since each overring is pseudo-valuation, and $\cap M_\alpha = c$ such that $c^{-1} \in F = \{V_\lambda\}_{\lambda\in\Lambda}$ for each $\lambda \in \Lambda$, then [3, Proposition 3.3] there exist $c \in \cap M_\alpha$ such that $c^{-1} \in V_\lambda$ for each $\lambda \in \Lambda$ being a pseudo-valuation. □

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Department of Mathematics and Statistics, Caledonian College of Engineering, PO Box 2322, CPO Seeb 111, Sultanate of Oman
E-mail address: sirwak2003@yahoo.com