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The structure of concept lattice based on matroidal approach

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Abstract

For a context, using an inducing bipartite graph, this paper provides a characterization for a concept and establishes a matroid. Further, with the assistance of this matroid, this paper seeks out the lattice structure of all the concepts with a hierarchical order. All these results shows the potential and merit in using matroidal approaches for designing and studying concept lattice.

Key words matroid; bipartite graph; concept lattice; structure

1 Introduction

Concept lattice, that is, Formal Concept Analysis (FCA), was first introduced in [1] and has grown to a powerful theory for data analysis, information retrieval and knowledge discovery (cf. [2-4]). FCA's increasing importance has many reasons (see [5, 6]). Among them, we see that the typical use is related to the structure of concepts in lattice and the information retrieval.

We should notice the following current situation and analysis for FCA.

(1.1) It is well known that constructing the lattice structure for all concepts in a context is important and interest for FCA.

(1.2) Matroids were first proposed in [7]. It has been found that matroids are effective and useful in concept lattice theory (see [8-11]).

(1.3) As everyone knows, the proof method of purely lattice theory for the basic theorem of FCA (i.e. Theorem 3 in [3, p.20]) causes the results and methods of lattice theory to apply in the research of FCA. If we obtain the basic theorem by the way of matroids, then we believe that matroids can be applied in the study of FCA as a fundamental tool. This is good for the study of both FCA and matroids.

(1.4) Motivated from different ways with graph theory, many authors research on the structure of concept lattices (see [12-15]). Combining [12], [3, p.20, Theorem 3] and [16, p.54, Theorem 2] with [16-18], we confirm that if we rely on graphs as a bridge between contexts and matroids, then the basic theorem may be obtained by matroidal approaches.

The main goal in this paper is that using matroidal ways proves Theorem 3 in [3, p.20]. To arrive at the goal, this paper will use bipartite graph model to search our matroidal approaches. The concrete process in this paper is that with the established bipartite graph for a context in [12], we will utilize the property of some subgraphs and establish the construction of a matroid. With the aid of matroids, we prove that the family of all the concepts in a context with a hierarchical order is a complete lattice.

Since these constructions come from the bipartite graph in [12] which can be seen as a visualized diagram. Hence, the main results in this paper are simpler. Furthermore, the approaches and consequences in this paper are accepted more easily.

We infer that the way in this paper is not the only way to obtain the basic theorem of FCA in the matroid framework, but we think our presentation is a better one. In addition, we maintain that the application provided in this paper sets a precedent for proving the basic theorem of FCA by other methods and not the purely lattice-theoretical methods.

The rest of the paper is organized as follows. Section 2 reviews some fundamental notations relative to concept lattices, matroids and graphs. In Section 3, with matroidal ideas, we obtain the structure of all the concepts with a hierarchical order for a context.

We declare that in this paper, the basic facts of lattice theory and poset theory are as discussed in [17]. Throughout, $MinA$ denotes the set of minimal elements in a poset A; $L_1 \cong L_2$ if the same two mathematical structures L_1 and L_2, e.g. two lattices L_1 and L_2, are isomorphic.

2 Preliminaries

This section introduces some basic facts of concept lattices, matroids and graphs.

2.1 Concept lattices

This subsection gives only a brief overview of the basic facts for concept lattices. For a more detailed description, please refer to [3].

Definition 2.1.1 (1) [3, p.17] A *context* (O, P, I) consists of two sets O and P and a relation I between O and P. The elements of O (of P) are called the *objects* (*attributes*) of (O, P, I) and I is called the *incidence relation* of (O, P, I). We write oIp or $(o, p) \in I$.

(2) [3, p.18] For $A \subseteq O$ and $B \subseteq P$, we define $A' := \{p \in P \mid oIp \text{ for all } o \in A\}$ and $B' := \{o \in O \mid oIp \text{ for all } p \in B\}$. A *concept* of (O, P, I) is a pair (A, B) with $A \subseteq O$, $B \subseteq P, A' = B$ and $B' = A$. We call A the *extent* (B the *intent*) of the concept (A, B).

(3) [3, pp.19-20] Let (A_1, B_1) and (A_2, B_2) be two concepts of (O, P, I). If $A_1 \subseteq A_2$ (which is equivalent to $B_2 \subseteq B_1$), then we write $(A_1, B_1) \leq (A_2, B_2)$. The relation $\leq$ is called the *hierarchical order* of the concepts. The set of all the concepts of (O, P, I) ordered in this way is denoted by $\mathcal{B}(O, P, I)$ and is called the *concept lattice* of (O, P, I).

In this paper, (O, P, I) denotes a context. The following statements are for (O, P, I).

(2.1.1) Let $\mathcal{B}_O(O, P, I) = \{X \mid (X, B) \in \mathcal{B}(O, P, I)$ for some $B \subseteq P\}$. The authors [3] indicate that if $\mathcal{B}_O(O, P, I)$ has the same hierarchical order as $\mathcal{B}(O, P, I)$, then there is $\mathcal{B}(O, P, I) \cong \mathcal{B}_O(O, P, I)$.

We still denote as $\mathcal{B}_O(O, P, I)$ if $\mathcal{B}_O(O, P, I)$ owns the same hierarchical order as $\mathcal{B}(O, P, I)$.

(2.1.2) Let $a \in O$ and $b \in P$ satisfy $(a, y) \notin I$ and $(x, b) \notin I$ for any $y \in P$ and $x \in O$. This paper does not consider the context with full rows and full columns which expressed in [3,p.24], and the above object a and attribute b.

(2.1.3) Considering (2.1.1) and (2.1.2), we may state that in this paper, $(\emptyset, P)$ and $(O, \emptyset)$ are the minimum and the maximum in $\mathcal{B}(O, P, I)$ respectively.

2.2 Matroids

Matroids will aid us in our discussions in this paper. Hence, in this subsection, we will recall notations and properties relative to matroids, and for more detail, we refer to [16, 18].

Definition 2.2.1 [16, p.7] A *matroid* M is a finite set S and a collection $\mathcal{I}$ of subsets of S (called *independent sets*) such that (i1)-(i3) are satisfied.

(i1) $\emptyset \in \mathcal{I}$;

(i2) If $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$;

(i3) If $X, Y \in \mathcal{I}$ with $|X| = |Y| + 1$, then there exists $x \in X \setminus Y$ such that $Y \cup x \in \mathcal{I}$.

A *base* of M is a maximal independent set.

Lemma 2.2.1 [16, pp.121-123] Let $M_1, \ldots, M_n$ be matroids on S. Let $\mathcal{I} = \{X \mid X = X_1 \cup \ldots \cup X_n; X_i \in \mathcal{I}(M_i), (1 \le i \le n)\}$. Then $\mathcal{I}$ is the collection of independent sets of a matroid $M_1 \vee \ldots \vee M_n$ on S.

2.3 Graphs

In this paper, for the definitions of *bipartite graph* and *complete bipartite graph*, please see [19, p.4]; for the definition of *subgraph* and *induced subgraph*, please refer to [19, p.9]; for the definition of *neighbour set*, please see [19, p.72].

We only review and discuss bits of notations and terminologies of graph theory. For more detail of graph theory, please refer to [19].

Some notations 2.3.1 Let G be a graph.

(1) The set of edges in G is denoted as $E(G)$; the set of vertices is in notation $V(G)$.

(2) If $V(G) = \emptyset$, then it is in notation $G = \emptyset$.

(3) $G[V^*]$ is an induced subgraph of G where $V^* \subseteq V(G)$ and $V^* \neq \emptyset$.

(4) The neighbor set of $x \in V(G)$ is in notation $N_G(x)$.

Sometimes, if it does not follow a confusion from the text, we denote $N_G(x)$ as $N(x)$.

Let $S \subseteq V(G)$. $N_G(S)$, simply $N(S)$, is $\{y \in V(G) \mid y \in N(x)$ for every $x \in S\}$.

(5) If G is simple and $e \in E(G)$ with u and v as its two connected vertices, then e is sometimes in notation uv.

From graph theory, we easily gain the following statements.

Lemma 2.3.1 Let G be a bipartite graph with $V(G) = X \cup Y$ satisfying $X \cap Y = \emptyset$. Then for $A, B \subseteq X$ (or $A, B \subseteq Y$), there are the following statements.

(1) $N(A) = \bigcap\limits_{a \in A} N(a)$. (2) $N(A \cup B) = N(A) \cap N(B)$. (3) $N(A) \subseteq N(B)$ if $B \subseteq A$.

To reformulate some results and search out some properties on FCA in matroid frameworks, we introduce a graph construction and obtain some properties on this construction.

Definition 2.3.1 [12] $G_{(O,P,I)}$, a *bipartite graph inducing from a context* (O, P, I), is $(O \cup P, \{(o, p) \mid oIp\})$, i.e. $V(G_{(O,P,I)}) = O \cup P$ and $E(G_{(O,P,I)}) = \{(o, p) \mid oIp\}$.

Lemma 2.3.2 $G_{(O,P,I)}$ has the following properties.

(1) $G_{(O,P,I)}$ is simple.

(2) $X' = N(X)$ for any $X \subseteq O$ (or $X \subseteq P$).

(3) If $X \subseteq O$ and $Y \subseteq P$, then $X = \emptyset \Rightarrow N(X) = P$; $Y = \emptyset \Rightarrow N(Y) = O$;

$X = O \Rightarrow N(X) = \emptyset$; $Y = P \Rightarrow N(Y) = \emptyset$.

Especially, if X is an extent and Y is an intent, then

$$N(X) = P \Rightarrow X = \emptyset;\ N(Y) = O \Rightarrow Y = \emptyset.$$

Proof (1)-(3) are straightforward from Definition 2.1.1 and Definition 2.3.1 with (2.1.3).

In fact, $X \subset O \Leftrightarrow N(X) \subset P$ is true for every contexts considered in this paper according to Subsection 2.1 and Definition 2.3.1.

Corollary 2.3.1 $(X, Y) \in \mathcal{B}(O, P, I) \Longleftrightarrow X = N(Y)$ and $Y = N(X)$ in $G_{(O,P,I)}$.

Proof Routine verification from Definition 2.1.1 and Lemma 2.3.2.

For clarity of exposition, in what follows, $G_{(O,P,I)}$ is sometimes in notation G if we find no confusion from text.

3 Structure of concept lattices

With the aid of an inducing bipartite graph of a context, this section, based on matroid ideas, characterizes a concept and proves that the collection of all the concepts with the hierarchical order is a complete lattice. In addition, we present the infinimum and the supremum for two concepts in the above complete lattice.

Lemma 3.1 Let $X \subseteq O$. Then $G[X \cup N(X)]$ is a complete bipartite graph.

Proof If $N(X) = \emptyset$ (or If $X = \emptyset$). Then $E(G[X \cup N(X)]) = \emptyset$. Thus, $G[X \cup N(X)]$ is a complete bipartite graph with $V(G[X \cup N(X)]) = X = O \neq \emptyset$ if $N(X) = \emptyset$ (or with $V(G[X \cup N(X)]) = N(X) = P \neq \emptyset$ if $X = \emptyset$).

Routine verification from the correspondent definition if $N(X) \neq \emptyset$.

To express easily, we give a notation:

Let $X, S \subseteq O$ and $Y, T \subseteq P$. If $G[X \cup Y]$ is a subgraph of $G[S \cup T]$ with $G[X \cup Y] \neq G[S \cup T]$, then this is denoted as $G[X \cup Y] \subset G[S \cup T]$.

Definition 3.1 Let $X \subseteq O$. If no $S \subseteq O$ satisfies $N(X) = N(S)$ and $G[X \cup N(X)] \subset G[S \cup N(S)]$, then $G[X \cup N(X)]$ is called a *maximal subgraph* in G.

Next, we characterize a concept.

Theorem 3.1 $(A, B) \in \mathcal{B}(O, P, I)$ if and only if $B = N(A)$ and $G[A \cup B]$ is maximal.

Proof ($\Rightarrow$) The needed result is evident from Lemma 2.3.2 and Corollary 2.3.1 if $(A, B) \in \{(O, \emptyset), (\emptyset, P)\}$.

If $(A, B) \in \mathcal{B}(O, P, I) \setminus \{(O, \emptyset), (\emptyset, P)\}$. Then $B = N(A)$ and $A = N(B)$ from Corollary 2.3.1. Suppose that there exists $S \subseteq O$ satisfying $N(S) = B$ and $G[A \cup B] \subset G[S \cup B]$. Then, by Lemma 2.3.2 and Lemma 3.1, $sb \in E(G)$ is correct for every $s \in S \setminus A$ and any $b \in B$. Moreover, $s \in A$ holds according to $A = N(B)$. This is a contradiction to $s \notin A$. Therefore, $G[A \cup B]$ is maximal.

($\Leftarrow$) Let A and B satisfy the given conditions.

If $B = P$ (or if $B = \emptyset$). By $B = N(A)$ and Lemma 2.3.2, $A = \emptyset$ holds (or $A = O$ holds). So, (A, B) is $(\emptyset, P) \in \mathcal{B}(O, P, I)$ (or (A, B) is $(O, \emptyset) \in \mathcal{B}(O, P, I)$).

If $B \subset P$ and $B \neq \emptyset$. $B = N(A)$ follows $A \subseteq N(B) = N(N(A))$. Thus $B = N(N(B))$ holds since $N(N(N(A))) = N(A)$. Suppose $d \in N(B) \setminus A \neq \emptyset$. Then, $dx \in E(G)$ is correct for any $x \in B$. This shows $G[A \cup B] \subset G[N(B) \cup B]$. Therefore, $G[A \cup B]$ is not maximal, a contradiction. Hence, we obtain $A = N(B)$. Using Corollary 2.3.1, (A, B) is a concept.

By Definition 2.1.1 and Theorem 3.1, we easily obtain the following consequence.

Corollary 3.1 If $S = N(N(T))$ for $T \subseteq O$. Then $G[S \cup N(T)]$ is maximal.

One of the main purposes in FCA is to find the structure among concepts. For this purpose, we need the following lemmas.

Lemma 3.2 For any $X \subseteq O$, $\mathcal{I}_X = \{Y \subseteq X \mid G[Y \cup N(Y)] \neq \emptyset\}$ is the family of independent sets of a matroid on O, and X is the unique base in the matroid $(O, \mathcal{I}_X)$.

Proof $N(\emptyset) = P$ follows $V(G[\emptyset \cup P]) = P \neq \emptyset$. So, $\emptyset \in \mathcal{I}_X$ holds. Let $Y \subseteq X$ and $Y \neq \emptyset$. Then $Y \in \mathcal{I}_X$ holds since $Y \subseteq V(G[Y \cup N(Y)])$ follows $G[Y \cup N(Y)] \neq \emptyset$.

Thus, $Y_1 \in \mathcal{I}_X$ holds no matter $Y_1 = \emptyset$ or $Y_1 \neq \emptyset$ if $Y_1 \subseteq Y_2 \in \mathcal{I}_X$.

Let $Z_1, Z_2 \in \mathcal{I}_X$ such that $|Z_1| = |Z_2|+1$. This implies that there exists $a \in Z_1 \setminus Z_2 \subseteq X$. Thus, we obtain $Z_2 \cup \{a\} \neq \emptyset$. By the above, this follows $Z_2 \cup \{a\} \in \mathcal{I}_X$.

The other needed results are evidently followed by Definition 2.2.1.

Lemma 3.3 Let $(X, N(X)), (Y, N(Y)) \in \mathcal{B}(O, P, I)$. Then $(X \cap Y, N(X \cap Y))$ is the infinimum of $(X, N(X))$ and $(Y, N(Y))$ in $\mathcal{B}(O, P, I)$.

Proof $X \cap Y \subseteq X, Y$ follows $N(X), N(Y) \subseteq N(X \cap Y)$ by Lemma 2.3.1(3). In addition, $X \cap Y \subseteq N(N(X \cap Y))$ is evident. Suppose $a \in N(N(X \cap Y)) \setminus X \cap Y$. Then, $ab_x, ab_y \in E(G)$ hold for any $b_x \in N(X)$ and $b_y \in N(Y)$. So, $a \in N(N(X))$ and $a \in N(N(Y))$ hold. However, $(X, N(X)), (Y, N(Y)) \in \mathcal{B}(O, P, I)$ and Corollary 2.3.1 together implies $X = N(N(X))$ and $Y = N(N(Y))$. Hence, we obtain $a \in X \cap Y$, a contradiction. Thus, we receive $X \cap Y = N(N(X \cap Y))$ and the maximality of $G[(X \cap Y) \cup N(X \cap Y)]$ by Corollary 3.1, we find $(X \cap Y, N(X \cap Y)) \in \mathcal{B}(O, P, I)$ by means of Theorem 3.1.

By poset theory, $\{\mathfrak{B}_Z \mid Z \subseteq O\}$ with the set inclusion is evidently a poset, where $\mathfrak{B}_Z$ is the set of bases in the matroid $(O, \mathcal{I}_Z)$. In fact, $\mathfrak{B}_Z = \{Z\}$ holds following Lemma 3.2. Obviously, $\mathfrak{B}_{X \cap Y}$ is the infinimum of $\mathfrak{B}_X$ and $\mathfrak{B}_Y$ in $(\{\mathfrak{B}_Z \mid Z \subseteq O\}, \subseteq)$. Additionally, up to the isomorphism, the poset $(\mathcal{B}_O(O, P, I), \leq)$ is evidently a subposet of $(\{\mathfrak{B}_Z \mid Z \subseteq O\}, \subseteq)$.

Considered $\mathcal{B}(O, P, I) \cong \mathcal{B}_O(O, P, I)$ and the above, we attain $(X \cap Y, N(X \cap Y))$ to be the infinimum of $(X, N(X))$ and $(Y, N(Y))$ in $\mathcal{B}(O, P, I)$.

For two concepts (A, B) and (C, D), under the hierarchical order "$\leq$", by $|\mathcal{B}(O, P, I)| < \infty$ and the properties in lattice theory, we obtain

$$(A, B) \vee (C, D) = \bigwedge_{(A,B),(C,D) \leq (A_j, B_j),\ (A_j, B_j) \in \mathcal{B}(O,P,I),\, j \in \mathfrak{J}} (A_j, B_j).$$

Furthermore, by Lemma 3.3, the extent of $(A, B) \vee (C, D)$ is $\bigcap_{j \in \mathfrak{J}} A_j$. Thus, the intents of $(A, B) \wedge (C, D)$ and $(A, B) \vee (C, D)$ are $N(A \cap C)$ and $N(\bigcap_{A,C \subseteq A_j \in \mathcal{B}_O(O,P,I),\, j \in \mathfrak{J}} A_j)$ respectively.

Therefore, we obtain the following theorem.

Theorem 3.2 Let $A, C \in \mathcal{B}_O(O, P, I)$. Then, with the hierarchical order,

(1) $\mathcal{B}_O(O, P, I)$ is a complete lattice. The supremum and infinimum are given by

$$A \vee C = \bigcap_{A,C \subseteq A_j \in \mathcal{B}_O(O,P,I),\, j \in \mathfrak{J}} A_j, \qquad A \wedge C = A \cap C.$$

(2) $\mathcal{B}(O, P, I)$ is a complete lattice.

Theorem 3.2(2) is the same as shown in [3] though the method in [3] is different from ours. This shows the success of matroidal approaches to prove the basic theorem for FCA.

Let $(X, N(X)), (Y, N(Y)) \in \mathcal{B}(O, P, I)$, Lemma 3.3 gives $(X, N(X)) \wedge (Y, N(Y))$ by a direct way. We hope to find $(X, N(X)) \vee (Y, N(Y))$ in a more direct way not as Theorem 3.2. Comparing Lemma 2.2.1 with Theorem 3.2, we conjecture that $X \cup Y$ is an extent since $M_X \vee M_Y = M_{X \cup Y}$. The following example shows the impossibility of this conjecture.

Example 3.1 Let $(\{1,2,3,4,5\},\{a,b,c\},I)$ be a context, where the set of objects is $\{1,2,3,4,5\}$; the set of attributes is $\{a,b,c\}$; the binary relation I is shown in Table 3.1.

	a	b	c
1	0	0	$\times$
2	$\times$	0	0
3	$\times$	0	0
4	0	$\times$	$\times$
5	0	$\times$	$\times$

Table 3.1 A context for the conjecture

The inducing graph of this context is shown in Figure 3.1. We see $N(1)=\{c\}, N(2)=N(3)=\{a\}, N(4)=N(5)=\{b,c\}, N(a)=\{2,3\}, N(b)=N(b,c)=\{4,5\}$ and $N(c)=\{1,4,5\}$. Thus, using Theorem 3.1, all the concepts are $(\emptyset, abc), (12345,\emptyset), (23,a), (45,bc)$ and $(145,c)$. The lattice structure of concepts is shown in Figure 3.2.

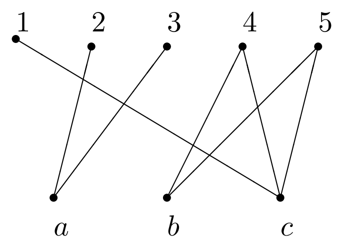

Figure 3.1
Inducing graph
for the context in Table 3.1

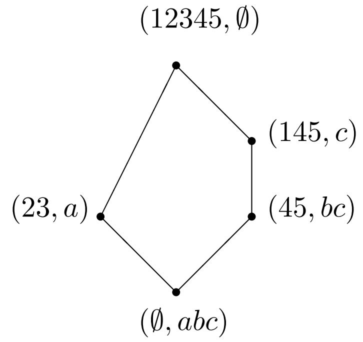

Figure 3.2
Concept lattice
for the context in Table 3.1

By Lemma 3.2, we obtain $\mathcal{I}_{\{2,3\}}=\{X\subseteq\{2,3\}\mid G[X\cup N(X)]\neq\emptyset\}, \mathcal{I}_{\{4,5\}}=\{X\subseteq\{4,5\}\mid G[X\cup N(X)]\neq\emptyset\}$ and $\mathcal{I}_{\{2,3,4,5\}}=\{X\subseteq\{2,3,4,5\}\mid G[X\cup N(X)]\neq\emptyset\}$. According to Lemma 2.2.1(2), there is $\mathcal{I}_{\{2,3,4,5\}}=\mathcal{I}_{\{2,3\}}\cup\mathcal{I}_{\{4,5\}}=\{X\cup Y\mid X\in\mathcal{I}_{\{2,3\}}, Y\in\mathcal{I}_{\{4,5\}}\}$. However, $N(\{2,3,4,5\})=\emptyset$ and $N(\emptyset)=\{1,2,3,4,5\}\neq\{2,3,4,5\}$. Thus, by Theorem 3.1, $\{2,3,4,5\}$ is not an extent.

By Corollary 2.3.1, we can write $(A,B)\in\mathcal{B}(O,P,I)$ as $(A,N(A))\in\mathcal{B}(O,P,I)$.

We will next seek a direct method to search out the supremum of two concepts.

Theorem 3.3 Let $(X,N(X)),(Y,N(Y))\in\mathcal{B}(O,P,I)$ and $S=N(N(X)\cap N(Y))$. Then $(X,N(X))\vee(Y,N(Y))=(S,N(X)\cap N(Y))\in\mathcal{B}(O,P,I)$.

Proof We may easily show $N(X)\cap N(Y)\subseteq N(X),N(Y)$, and so $N(N(X)),N(N(Y))\subseteq N(N(X)\cap N(Y))$ since Lemma 2.3.1. That is, $N(N(X)),N(N(Y))\subseteq S$ holds. In view of Corollary 2.3.1 and $X,Y\in\mathcal{B}_O(O,P,I)$, we receive $X=N(N(X))$ and $Y=N(N(Y))$. Thus, it follows $X,Y\subseteq S$. Considering Lemma 2.3.1, we find $N(S)\subseteq N(X),N(Y)$, and so $N(S)\subseteq N(X)\cap N(Y)$. Additionally, $S=N(N(X)\cap N(Y))$ implies $N(X)\cap N(Y)\subseteq N(S)$.

Combining the above, it follows $N(S)=N(X)\cap N(Y)$. Therefore, $(S,N(X)\cap N(Y))\in\mathcal{B}(O,P,I)$ by Corollary 2.3.1.

From the above proof, we know $X, Y \subseteq S$. This means $(X, N(X)), (Y, N(Y)) \leq (S, N(X) \cap N(Y))$ in $\mathcal{B}(O, P, I)$.

Let $(A, B) \in \mathcal{B}(O, P, I)$ such that $(A, B) = (X, N(X)) \vee (Y, N(Y))$. This implies $X, Y \subseteq A$ and $A \subseteq S$. Thus, we obtain $N(A) \subseteq N(X), N(Y)$. Furthermore, in view of Lemma 2.3.1 and Corollary 2.3.1, it follows $B = N(A) \subseteq N(X) \cap N(Y)$, and so $N(N(X) \cap N(Y)) \subseteq N(B) = A$. That is, $S \subseteq A$ holds. Therefore, it yields out $A = S$.

Thus, $(S, N(X) \cap N(Y))$ is the supremum of $(X, N(X))$ and $(Y, N(Y))$ in $\mathcal{B}(O, P, I)$.

We will use the above results in this section to obtain some results for protoconcepts which is produced in [20] as a new content of FCA.

The authors [21] indicate that FCA has been formally enriched by introducing the notions of semiconcept and protoconcept. Wille proves [20] that a semiconcept is a protoconcept. Hence, we will only pay attention to protoconcepts. A *protoconcept* of (O, P, I) is defined as a pair (A, B) with $A \subseteq O$ and $B \subseteq P$ such that $A' = B''$, which is equivalent to $B' = A''$ (see [20, 21]). In addition, Wille describes [20] that the set of all protoconcepts of (O, P, I) is structured by the order $\sqsubseteq$ which is defined by $(A_1, B_1) \sqsubseteq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2$ and $B_1 \supseteq B_2$. Let $\mathscr{B}(O, P, I)$ be the set of all the protoconcepts of (O, P, I) with $\sqsubseteq$.

Let $(A_j, B_j) \in \mathcal{B}(O, P, I), (j = 1, 2)$. We may easily find that

(R1) $(A_1, B_1) \sqsubseteq (A_2, B_2) \Leftrightarrow (A_1, B_1) \leq (A_2, B_2)$;

(R2) $(A, B) \in \mathcal{B}(O, P, I) \Rightarrow (A, B) \in \mathscr{B}(O, P, I)$.

We claim that a protoconcept is perhaps not a concept. Suppose that every protoconcept is a concept. Then Theorem 3.2, (R1) and (R2) taken together implies that $\mathscr{B}(O, P, I)$ is a complete lattice. However, the authors [20, 21] points that in general, $\mathscr{B}(O, P, I)$ does not yield a lattice structure. These discussions follows a contradiction. Therefore, we are assured the correct of the claim.

Though our method here used Theorem 3.2 and graph background for this claim is different from that in [21], both methods reveal the same reality. Additionally, following the approach in the proof of the basic theorem on concepts in [3], the authors [21] give the basic theorem on protoconcept algebras. We hope that following the method in the proof of Theorem 3.2, we will provide a new way to prove the basic theorem on protoconcept algebras. Furthermore, we can compare more connections and differences between protoconcepts and concepts by matroidal approach in order to enrich FCA and its applications.

In the future works, with matroidal approaches, for a context, we will study on how to search out the concept lattice, and explore the wider applications of concept lattices.

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