Application of Generalized Probability Weighted Moments for Skew Normal Distribution

E. A. Elsherpieny[1], Amal S. Hassan[2] and Neema M. El Haroun[3]

[1] Department of Mathematical Statistical, Institute of Statistical Studies & Research, Cairo University, (Gize, Egypt)

[2] Department of Mathematical Statistical, Institute of Statistical Studies & Research, Cairo University, (Gize, Egypt)

[3] PhD. Student, Institute of Statistical Studies & Research, Cairo University, (Gize, Egypt)

ABSTRACT— ***Skewness is often present in a wide range of applied problems. One possible approach to model this skewness is based on the class of skew normal distributions. This article focuses on estimating the unknown parameters of skew normal distribution. Generalized probability weighted moments, probability weighted moments and fractional moments estimating methods are investigated in skew normal distribution. Comparison between estimators is made through Monte Carlo simulation via their mean square errors. Comparison study revealed that the fractional moment estimators are better than generalized probability weighted moment and probability weighted moment estimators***

Keywords— Probability weighted Moments, Generalized probability weighted Moments, Fractional moments and Skew normal distribution.

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1. INTRODUCTION

The normal distribution is popular and easy to handle, but also is not always adequate to insurance or finance application. Azzalini [3] introduced a new class of normal distribution called skew normal (SN) distribution. The SN distribution includes the normal distribution as a special case. This family of distributions has a shape parameter α that defines the direction of the asymmetry of the distribution, also called skewness parameter. The two parameters SN model has the following density function;

$$f(x,\mu,\alpha) = 2\ \phi(x-\mu)\ \ \Phi[\alpha(x-\mu)],\ x \in R. \tag{1}$$

The corresponding cumulative distribution function is given by;

$$F(x,\mu,\alpha) = \int_{-\infty}^{x} 2\ \phi(x-\mu)\ \ \Phi[\alpha(x-\mu)],\ x \in R, \tag{2}$$

where, $\mu \in R$ is location parameter and $\alpha \in R$ represents the shape parameter. The notations $\phi(x-\mu)$ and $\Phi[\alpha(x-\mu)]$ denote the density and cumulative distribution functions of the two parameters of SN distribution, respectively. The SN distribution reduced to normal distribution for $\alpha = 0$.

Estimation of skewness parameter using maximum likelihood method studied by many authors (see for example Pewsey [13], Gupta and Brown [8]). Bayes and Branco [4] showed the Bayesian approach using Monte Carlo methods is a good alternative to make inference under the skewness parameter. They provided an approximation for the presented to compare the bias, mean square error (MSE) and interval estimates using the maximum likelihood and different Bayes estimator. Flecher et al [6] computed and estimated the parameter of SN distribution by the probability weighted moments (PWMs) method. They mention that the computation of PWMs seem very difficult to derive because the cumulative of SN is very complex. Flecher et al [6] replaced the cumulative of skewness by the cumulative Gaussian distribution for greatly simplifies the problem. Comparing the results of PWMs and maximum likelihood methods, the PWMs gave the smallest MSE values.

The generalized probability weighted moments (GPWMs) were introduced by Rasmussen [14] as a tool for estimating the parameters of probability distribution expressible in inverse form. Rasmussen [14] estimated the unknown parameters of generalized Pareto distribution using GPWMs, PWMs and traditional moments methods. It is noted that the GPWMs estimators gave the smallest bias and MSE. Ashkar and Mahdi [2], developed the GPWMs method for log logistic distribution. They concluded that the estimators based on GPWMs are smaller than the corresponding estimators based on PWMs and maximum likelihood methods, especially for small samples.

The GPWMs, generalized moments and maximum likelihood methods of estimation are investigated in two parameters Weibull distribution by Mahdi and Ashkar[12]. Point estimators for positive and negative shape parameters and for quantile are derived. The performance of the three estimating methods is given through simulation. The results show that the GPWMs method may in some situation lead to a slight gain in quantile estimation accuracy. Furthermore, they concluded that maximum likelihood method is the most recommendable one. Recently, El Haroun [5] used the GPWMs method to estimate the parameters of the generalized exponential distribution and obtain Asymptotic variance of the estimators.

This article deals with estimating the unknown parameters of SN distribution based on GPWMs, PWMs and fractional moments (FMs) methods. Monte Carlo simulation is performed to compare the performance of different estimators.

This paper is organized as follow. In Section 2, the GPWMs estimators for the shape and location parameters for SN distribution will be derived. The estimates of parameters for SN distribution are developed using PWMs method in section 3. The estimates of unknown parameters for SN distribution based on FMs will be discussed in Section 4. Simulation results and discussions are contained in Section 5. Finally, conclusions are included in Section 6. Tables and some Figures are included in the appendix.

2. THE GPWMs METHOD FOR SN DISTRIBUTION

The GPWMs method has been used in estimating the parameters of many distributions that can be expressible in inverse form. According to Rasmussen [14], the GPWMs take the following form;

$$M_{p,u,v} = E\left(X^{p}\{F(X)\}^{u}\{1-F(X)\}^{v}\right), \tag{3}$$

where X is a random variable and $F(X)$ is cumulative distribution function. The common practice has been to consider $p=1$ and , u and v to be real values (i.e. exponents u and v of the GPWMs can be ratio or integers).

The GPWMs of order $p=1$ and $v=0$, is given by $M_{1,u,0}$ as the following;

$$M_{1,u,0} = E\left(X\left(F(X)\right)^{u}\right) = \int_{-\infty}^{\infty} x\left(\int_{-\infty}^{x} f(t)\,dt\right)^{u} f(x)\,dx\ , \tag{4}$$

where, $f(x)$ and $f(t)$ are the probability functions of the random variable X .

In particular, the formula (4) will be used, with $u = u_1, u_2$ then $M_{1,u,0}$ takes the following forms,

$$M_{1,u_1,0} = E\left(X\left(F(X)\right)^{u_1}\right) = \int_{-\infty}^{\infty} x\left(\int_{-\infty}^{x} f(t)\,dt\right)^{u_1} f(x)\,dx, \tag{5}$$

$$M_{1,u_2,0} = E\left(X\left(F(X)\right)^{u_2}\right) = \int_{-\infty}^{\infty} x\left(\int_{-\infty}^{x} f(t)\,dt\right)^{u_2} f(x)\,dx. \tag{6}$$

To obtain the theoretical GPWMs for SN distribution substitute probability density (1) in formulas (5) and (6) therefore;

$$M_{1,u_1,0} = \int_{\mu}^{\infty} x\left[\int_{\mu}^{x} 2\,\phi(t-\mu)\,\Phi\{((t-\mu)\alpha)\}dt\right]^{u_1} \left[2\,\phi(x-\mu)\,\Phi\{((x-\mu)\alpha)\}\right] dx\,. \tag{7}$$

$$M_{1,u_2,0} = \int_{\mu}^{\infty} x\left[\int_{\mu}^{x} 2\,\phi(t-\mu)\,\Phi\{((t-\mu)\alpha)\}dt\right]^{u_2} \left[2\,\phi(x-\mu)\,\Phi\{((x-\mu)\alpha)\}\right] dx \tag{8}$$

Let $x_{(1)}, x_{(2)},..., x_{(n)}$ be a random sample of size n from the distribution function $F(X)$, and $x_{(1)} < x_{(2)},... < x_{(n)}$ be the corresponding ordered sample. Hosking et al [10] proposed sample GPWMs, denoted by $\hat{M}_{1,u,v}$, as the following;

$$\hat{M}_{1,u,v} = \sum_{i=1}^{n} x_{(i)} p_i^u (1-p_i)^v, \tag{9}$$

where, $x_{(i)}$ is the i^{th} observation in the ordered sample and $p_i = \dfrac{i-0.35}{n}$.

To obtain the sample GPWMs for SN distribution; substitute; $u = u_1$ and $u = u_2$ after setting $v = 0$ in (9) therefore;

$$\hat{M}_{1,u_1,0} = \frac{1}{n}\sum_{i=1}^{n} x_i \left(\frac{i-0.35}{n}\right)^{u_1}, \tag{10}$$

and $$\hat{M}_{1,u_2,0} = \frac{1}{n}\sum_{i=1}^{n} x_i \left(\frac{i-0.35}{n}\right)^{u_2}. \tag{11}$$

Therefore, the GPWMs estimator of α and μ denoted by $\hat{\alpha}_g$ and $\hat{\mu}_g$ can be obtained by equating the sample GPWMs in (10) and (11) with the population GPWMs in (7) and (8) as the following;

$$\int_{\mu}^{\infty} x \left[\int_{\mu}^{\infty} 2\,\phi(t-\mu)\,\Phi\{((t-\mu)\alpha)\}dt\right]^{u_1} [2\,\phi(x-\mu)\,\Phi\{((x-\mu)\alpha)\}]\,dx = \frac{1}{n}\sum_{i=1}^{n} x_i\left(\frac{i-0.35}{n}\right)^{u_1}, \tag{12}$$

$$\int_{\mu}^{\infty} x \left[\int_{\mu}^{\infty} 2\,\phi(t-\mu)\,\Phi\{((t-\mu)\alpha)\}dt\right]^{u_2} [2\,\phi(x-\mu)\,\Phi\{((x-\mu)\alpha)\}]\,dx = \frac{1}{n}\sum_{i=1}^{n} x_i\left(\frac{i-0.35}{n}\right)^{u_2}. \tag{13}$$

The GPWMs estimator $\hat{\alpha}_g$ and $\hat{\mu}_g$ can be obtained by solving numerically the non linear equations using (12) and (13) through Monte Carlo simulation.

3. THE PWMs METHOD FOR SN DISTRIBUTION

Greenwood et al [7], proposed a class of moments called PWMs. The method of PWMs is used for estimating the parameters of distributions that are analytically expressible only in inverse forms such as, Tukey's Lambda and Wakeby distributions which are potentially useful to flood frequency analysis. The PWMs are formally defined as equation (3). The exponents of PWMs method take any real values but most of application takes the exponents (u, v) integers only.

Song and Ding [15] set another formula for PWMs to estimate the unknown parameters of distributions that cannot be expressible in an inverse form. They rewrite the formula of PWMs by substituting $dF = f(x)dx$ in (3) as the following;

$$M_{p,u,v} = \int_{-\infty}^{\infty} x^p \left(\int_{-\infty}^{x} f(t)dt\right)^u \left(1 - \int_{-\infty}^{x} f(t)\right)^v f(x)\,dx,$$

where $f(x)$ and $f(t)$ are the probability functions of the random variable X.

To obtain the theoretical PWMs estimator for the SN distribution substitute; $u_1 = 0$ and $u_2 = 1$ in (7) and (8), therefore;

$$M_{1,0,0} = \int_{\mu}^{\infty} x\,[2\,\phi(x-\mu)\,\Phi\{((x-\mu)\alpha)\}]\,dx, \tag{14}$$

$$M_{1,1,0} = \int_{\mu}^{\infty} x \left[\int_{\mu}^{x} 2\,\phi(t-\mu)\,\Phi\{((t-\mu)\alpha)\}dt\right] [2\,\phi(x-\mu)\,\Phi\{((x-\mu)\alpha)\}]\,dx. \tag{15}$$

Let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be random sample of size n from the distribution function $F(X)$, and $x_{(1)} < x_{(2)}, \ldots < x_{(n)}$ be the corresponding ordered sample. Landwehr et al [11] proposed an unbiased estimator of PWMs say $\hat{M}_u$ as the following;

$$\hat{M}_u = n^{-1}\sum_{j=1}^{n}\frac{(j-1)(j-2)\ldots(j-r)}{(n-1)(n-2)\ldots(n-r)}x_j \ . \tag{16}$$

In this study, the following sample PWMs will be used;

$$\hat{M}_0 = \frac{1}{n}\sum_{i=1}^{n} x_i \ , \tag{17}$$

and $$\hat{M}_1 = \frac{1}{n(n-1)}\sum_{i=1}^{n}(i-1)x_i \ . \tag{18}$$

Therefore, the PWMs estimator of α and μ ,denoted by, $\hat{\alpha}_p$ and $\hat{\mu}_p$ can be obtained by equating the sample PWMs with the population PWMs as the following;

$$\int_{\mu}^{\infty} x\,[2\,\phi(x-\mu)\,\Phi\{((x-\mu)\alpha)\}]\,dx = \frac{1}{n}\sum_{i=1}^{n} x_i \ , \tag{19}$$

$$\int_{\mu}^{\infty} x\left[\int_{\mu}^{x} 2\,\phi(t-\mu)\,\Phi\{((t-\mu)\alpha)\}dt\right]\,[2\,\phi(x-\mu)\,\Phi\{((x-\mu)\alpha)\}]\,dx = \frac{1}{n(n-1)}\sum_{i=1}^{n}(i-1)x_i \ . \tag{20}$$

To get accurate results, PWMs estimators $\hat{\alpha}_p$ and $\hat{\mu}_p$ can be solved numerically in equations (19) and (20) by using Monte Carlo simulation.

4. THE FMs FOR SN DISTRIBUTION

The method of moment is one of the oldest methods of estimating the parameters of distribution. It is summarized in constructing estimators of the parameters which is based on matching the sample moments with the corresponding population moments. The exponents of tradition moments always take the integers values but it may not exist.

It is known that the sampling variability of moments increases with the order of the moments. Therefore in the traditional method of moments for estimation, lower order moments are used. To decrease the sampling variability and thereby to increase the efficiency of moment estimates, one can think of decreasing the order of moments to less than one. This idea of lowering the order of moments and thereby reducing the sampling variability of sample moments was proposed by Abdul Khalique [1]. This new method is known as fraction moments and is proposed for analyzing the multiplicity distributions of particles in inelastic processes. This method is based on the use of non integer moments of distribution. In general, this method can be used to analyze any distributions which are encountered in physics or mathematics. According to Adul Khalique [1], the l^{th} FMs of a random variable X with density function $f(x)$ takes the following form;

$$FM_l = \int_{0}^{\infty} x^l f(x)\,dx, \qquad 0 < l < 1. \tag{21}$$

For estimating the two parameters of SN distribution, the population FMs is obtained by setting $l = l_1$ and $l = l_2$ in equation (21), therfore;

$$FM_{l_1}(x,\alpha,\mu) = 2\int_{0}^{\infty} x^{l_1}\ \phi(x-\mu)\ \ \Phi[\alpha(x-\mu)], \tag{22}$$

$$FM_{l_2}(x,\alpha,\mu) = 2\int_{0}^{\infty} x^{l_2}\ \phi(x-\mu)\ \ \Phi[\alpha(x-\mu)]. \tag{23}$$

According to Abdul Khalique [1], the corresponding empirical l^{th} FMs from a random sample $x_1, x_2 \ldots x_n$ takes the following formula;

$$m_l = \frac{1}{n}\sum_{i=1}^{n} x_i^l \qquad 0 < l \ < 1 \tag{24}$$

To obtain the sample FMs for SN distribution substitute $l = l_1$ and $l = l_2$ in equation (24), therefore;

$$m_{l_1} = \frac{1}{n}\sum_{i=1}^{n} x_i^{l_1} , \qquad 0 < l_1 < 1 \tag{25}$$

$$m_{l_2} = \frac{1}{n}\sum_{i=1}^{n} x_i^{l_2} . \qquad 0 < l_2 < 1 \tag{26}$$

Hence, the FMs estimator of α and μ denoted by $\hat{\alpha}_f$ and $\hat{\mu}_f$ can be obtained by equating the sample FMs given in (25) and (26) with the corresponding population FMs given in (22) and (23) as the following;

$$2\int_0^\infty x^{l_1}\ \phi(x-\mu)\ \Phi[\alpha(x-\mu)] = \frac{1}{n}\sum_{i=1}^{n} x_i^{l_1} , \tag{27}$$

$$2\int_0^\infty x^{l_2}\ \phi(x-\mu)\ \Phi[\alpha(x-\mu)] = \frac{1}{n}\sum_{i=1}^{n} x_i^{l_2} . \tag{28}$$

To avoid complex ways of solution, FMs estimators $\hat{\alpha}_f$ and $\hat{\mu}_f$ can be worked out numerically in (27) and (28) using Monte Carlo simulation.

5. SIMULATION STUDIES OF SN DISTRIBUTION

Monte Carlo simulation has been performed to investigate the properties of the GPWMs, PWMs and FMs estimators for SN. An extensive simulation study is performed to compare the performance of the different methods of estimation mainly in terms of their MSE. The steps of simulation can be summarized as the following;

Step (1): 1000 random sample, $x_1, x_2, ..., x_n$, of sizes *n* = 5, 10, 15, 20, 25, 30, 35and 50 are generated from the SN distribution. This can be achieved by generating a random sample from closed form of SN that proposed by Henz [9] as follow;

$$Q_{i,j} = \left(\frac{\alpha}{\sqrt{1+\alpha^2}}\right).\left|L_{i,j}\right| + \left(\frac{1}{\sqrt{1+\alpha^2}}\right).W_{i,j} , \qquad for \quad i = 1,2..... \quad j = 1,2,.....$$

where $(w_1, w_2, ..., w_n)$ and $(L_1, L_2, ..., L_n)$are random sample from standard normal distribution.

Step (2): The true parameter selected values for the shape parameter α are 0 (0.3) 0.9 and the values for location parameter μ are 0.2 (0.3) 0.5. Choosing the exponent values of GPWMs ratio as $u_1 = 0.5$ and $u_2 = 1.5$. The exponent values of FMs are ratio between [0,1] as $u_1 = 0.89$ and $u_2 = 0.15$.

Step (3): For each combination of values of sample size n, α and μ, the parameters of distribution are estimated using three different estimation methods; GPWMs , PWMs and FMs.

Step (4): GPWMs estimators for α and μ denoted by $\hat{\alpha}_g$ and $\hat{\mu}_g$ for SN distribution is obtained by solving numerically the non-linear equations (12) and (13). By similar way, PWMs estimators $\hat{\alpha}_p$ and $\hat{\mu}_p$ of SN distribution are obtained by solving numerically equations (19) and (20). Also, FMs estimators $\hat{\alpha}_f$ and $\hat{\mu}_f$ of SN distribution are obtained by solving the non-linear equations (27) and (28).

Step (5): The MSE for the different estimators of the two parameters and for all sample sizes are Tabulated and represented through some Figures.

All simulation studies presented here are obtained via the MathCAD (2001) software. The MSE of the different estimators of α and λ are reported in Tables (1) and (2), respectively and shown in Figures (1-8). From simulation many observations can be made on the performance of the methods GPWM, PWMs and FMs. These observations are summarized as follows:

1. It is observed that the MSE of estimator for α decreases as the sample size increases. This indicate that all the methods provided consistent estimators for α [see Figures (1- 4)].
2. It is observed that the MSE of estimator for μ decreases as the sample size increases. This indicate that all the methods provided consistent estimators for μ [see Figures (5-8)].
3. Considering the MSE of the different estimators of α , it is clear from Table (1) that the FMs estimator has the minimum MSE in almost all of the cases considered for estimating α . [see Figures (1-4)].

4. Considering the MSE of the different estimators of μ, it is clear from Table (2) that the FMs estimator has the minimum MSE in almost all of the cases considered for estimating μ [see Figures (5-8)].

6. CONCLUSION

In this study, the GPWMs method has been applied for estimating the parameters of SN distribution as a skewness distribution. In more details, the performance of the methods of GPWM was compared to the performance of two methods, namely; the methods of PWMs and FMs. The comparative study revealed that the FMs works the best in almost all cases considered with respect to MSE. The performance of the PWMs method is the worst in terms of MSE. In addition, all the methods considered provide us with consistent estimators for α and λ.

7. REFERENCES

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APPENDIX

The next Figures represent MSE of shape and scale parameters estimator for SN distribution based on GPWMs, PWMs, and FMs

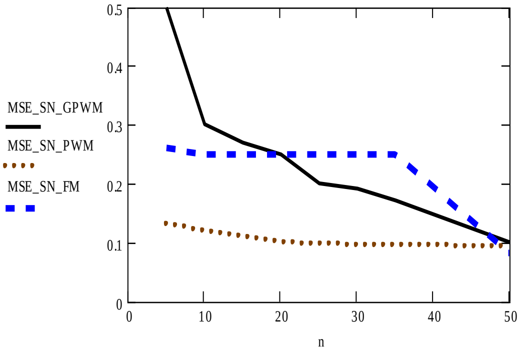

Figure 1: The MSE of SN distribution based on GPWMs, PWMs and FMs for $\alpha = 0$.

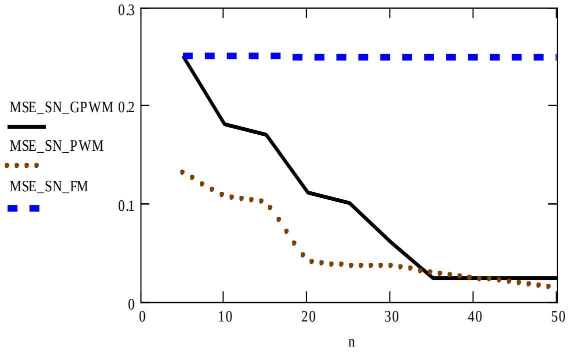

Figure 2: The MSE of SN distribution based on, GPWMs, PWMs and FMs for $\alpha = 0.3$.

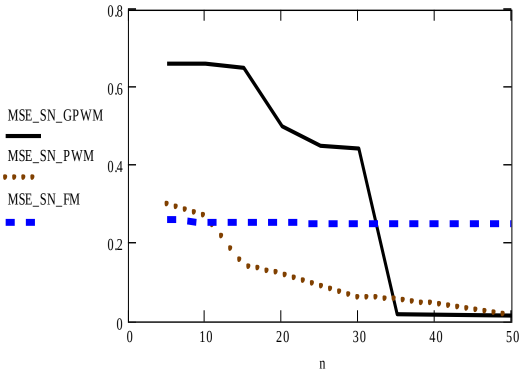

Figure 3: The MSE of SN distribution based on GPWMs, PWMs and FMs for $\alpha = 0.6$.

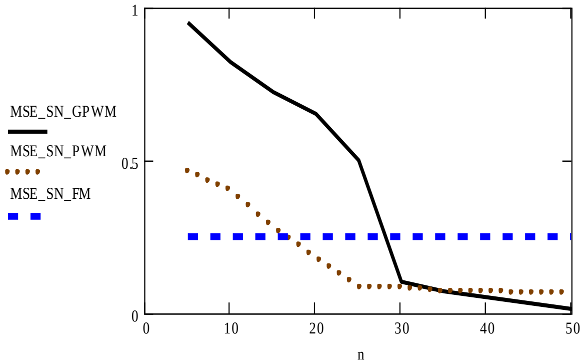

Figure 4: The MSE of SN distribution based on GPWMs, PWMs and FMs for $\alpha = 0.9$.

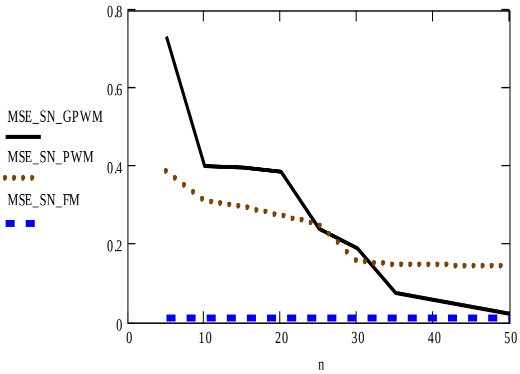

Figure 5 : The MSE of SN distribution based on GPWMs, PWMs and FMs for $\mu = 0.5$.

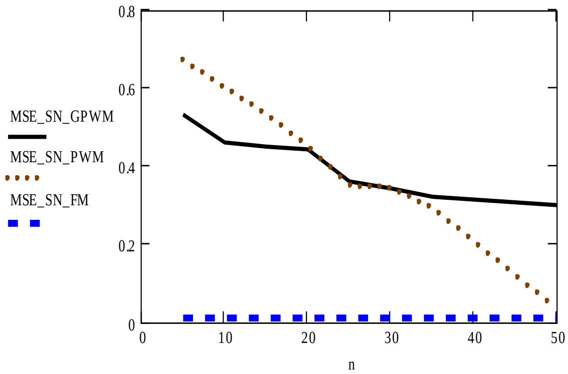

Figure 6: The MSE of SN distribution based on GPWMs, PWMs and FMs for $\mu = 0.4$.

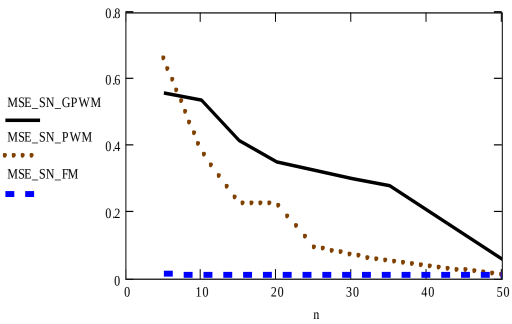

Figure 7: The MSE of SN distribution based on GPWMs, PWMs and FMs for $\mu = 0.3$.

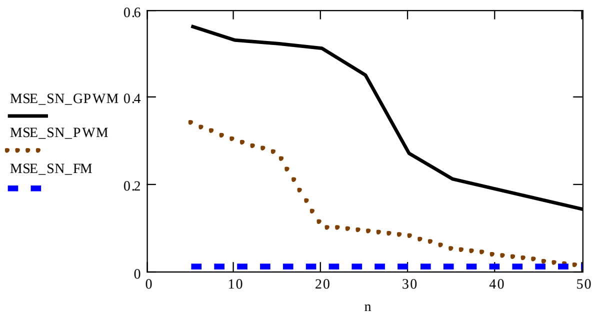

Figure 8: The MSE of SN distribution based on GPWMs, PWMs and FMs for $\mu = 0.2$.

Table (1): Mean and MSE for different value of the estimated shape parameter for SN distribution based on GPWMS, PWMs and FMs methods of estimation.

Sample size n	Properties Of α	GPWMs of $\hat{\alpha}_g$				PWMs of $\hat{\alpha}_p$				FMs of $\hat{\alpha}_f$			
		$\alpha=0$	$\alpha=0.3$	$\alpha=0.6$	$\alpha=0.9$	$\alpha=0$	$\alpha=0.3$	$\alpha=0.6$	$\alpha=0.9$	$\alpha=0$	$\alpha=0.3$	$\alpha=0.6$	$\alpha=0.9$
5	Mean	1.2109	0.0009	-0.4762	-0.3135	0.1390	0.1270	-0.0805	-0.0655	-0.0032	0.0002	0.00286	0.0043
	MSE	0.5054	0.2509	0.9529	0.6618	0.1303	0.1391	0.4631	0.3198	0.2635	0.2502	0.2553	0.2642
10	Mean	-0.0499	0.0712	-0.4237	-0.3135	0.1503	0.1721	-0.0387	-0.0228	0.00005	0.0002	0.0001	0.0001
	MSE	0.3024	0.1838	0.8532	0.6618	0.1223	0.1075	0.4079	0.2733	0.2500	0.2502	0.2499	0.2501
15	Mean	0.0211	0.0814	-0.3478	0.3101	0.1567	0.1819	0.0641	0.1177	0.00005	-0.0002	0.0001	0.0016
	MSE	0.2716	0.1752	0.7188	0.6562	0.1178	0.1012	0.2871	0.1461	0.2499	0.2502	0.2499	0.2510
20	Mean	0.0172	0.1541	-0.3111	1.2096	0.1813	0.2822	0.1744	0.1444	0.0001	0.0008	0.0002	0.00005
	MSE	0.2531	0.1196	0.6579	0.5036	0.1015	0.0474	0.1811	0.1269	0.2499	0.2499	0.2498	0.2500
25	Mean	0.0506	0.1687	-0.2294	-0.1721	0.1875	0.3110	0.3132	0.2002	0.0001	0.0001	0.0007	0.0009
	MSE	0.2020	0.1098	0.5320	0.4517	0.0976	0.0357	0.0822	0.0898	0.2499	0.2499	0.2498	0.2499
30	Mean	0.9413	0.2526	0.1440	-0.1662	0.1927	0.3122	0.3135	0.2569	0.0008	0.0002	0.0006	0.0009
	MSE	0.1948	0.0612	0.1267	0.4438	0.0944	0.0353	0.0821	0.0591	0.2498	0.2499	0.2498	0.2499
35	Mean	0.0779	0.3466	0.2219	0.6288	0.1952	0.5275	0.3418	0.2654	0.0002	0.0002	0.0005	0.0002
	MSE	0.1782	0.0235	0.0774	0.0166	0.0929	0.0297	0.0667	0.5502	0.2498	0.2498	0.2497	0.2498
50	Mean	0.1727	0.6528	0.3793	0.1617	0.1965	0.3840	0.3457	0.6176	0.7881	0.0003	0.0005	0.00007
	MSE	0.1071	0.0233	0.0146	0.0136	0.0944	0.0135	0.0647	0.0138	0.0830	0.2498	0.2497	0.2498

Table (2): MSE for different value of the estimated location parameter for SN distribution based on GPWMS, PWMs and FMs methods of estimation.

Sample size n	Properties Of μ	GPWMs of $\hat{\mu}_G$				PWMs of $\hat{\mu}_P$				FMs of $\hat{\mu}_F$			
		$\mu=0.5$	$\mu=0.4$	$\mu=0.3$	$\mu=0.2$	$\mu=0.5$	$\mu=0.4$	$\mu=0.3$	$\mu=0.2$	$\mu=0.5$	$\mu=0.4$	$\mu=0.3$	$\mu=0.2$
5	Mean	0.9564	0.8309	0.8344	0.8496	0.6559	0.9174	0.9134	0.6836	0.0006	0.0008	0.0011	0.0007
	MSE	0.7334	0.5342	0.5540	0.5619	0.3090	0.6682	0.6616	0.3405	0.0102	0.0105	0.0110	0.0104
10	Mean	0.7321	0.7756	0.8310	0.8254	0.6413	0.8773	0.7200	0.6433	0.0005	0.0007	0.0008	0.0006
	MSE	0.3996	0.4564	0.5340	0.5262	0.2930	0.6020	0.3849	0.2951	0.0102	0.0104	0.0104	0.0102
15	Mean	-0.5282	0.7697	0.7411	0.8221	0.6195	0.7741	0.5740	0.6225	0.0005	0.0006	0.0006	0.0006
	MSE	0.3946	0.4485	0.4110	0.5215	0.2699	0.4544	0.2247	0.2730	0.0101	0.0103	0.0103	0.0102
20	Mean	0.7198	0.7652	0.6913	0.8135	0.5989	0.6932	0.5722	0.4119	0.0005	0.0006	0.0006	0.0005
	MSE	0.3841	0.4425	0.3497	0.5091	0.2484	0.3519	0.2230	0.0973	0.0101	0.0102	0.0103	0.0102
25	Mean	0.5884	0.6972	0.6692	0.7693	0.7198	0.6879	0.5248	0.4047	0.0003	-0.0004	0.0003	0.0004
	MSE	0.2385	0.3567	0.3240	0.4480	0.3841	0.3457	0.1805	0.0928	0.0100	0.0102	0.0100	0.0101
30	Mean	0.5310	0.6816	0.6457	0.6186	0.4895	0.6733	0.4868	0.3758	0.0003	0.0005	0.0003	-0.0001
	MSE	0.1857	0.3383	0.2978	0.2689	0.1517	0.5286	0.1497	0.0761	0.0100	0.0101	0.0100	0.0100
35	Mean	0.3696	0.6633	0.6266	0.5641	0.4821	0.6462	0.4810	0.3261	0.0003	0.0004	0.0003	0.0003
	MSE	0.0727	0.3173	0.2773	0.2154	0.1460	0.2983	0.1451	0.0511	0.0100	0.0101	0.01003	0.0100
50	Mean	0.2459	0.6471	-0.1348	-0.2742	0.4752	0.2657	0.2930	0.2206	0.0002	0.0001	0.0003	0.0001
	MSE	0.0213	0.2993	0.0551	0.1400	0.1408	0.0275	0.0372	0.0145	0.0100	0.0100	0.0100	0.0100