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Tridiagonal Matrices with Permanent Values Equal to *k*-Jacobsthal Sequence

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ABSTRACT— ***We provide a proof that the permanents of certain tridiagonal matrices are natural numbers in a k-Jacobsthal sequence. As a consequence, such matrices are convertible.***

Keywords— Permanent, *k*-Jacobsthal Sequence, Convertible Matrix

1. INTRODUCTION

There are several studies on representing famous sequences of natural numbers as permanent of matrices. We start the article by discussing a few major ones. In [9] and [10], Bozkurt and Yilmaz consider one type of an $n\times n$ upper Hessenberg matrix of odd order

$$H_n = \begin{bmatrix} 1 & 1 & -1 & & & & & \\ 1 & 1 & 1 & 1 & & & 0 & \\ & 1 & 1 & 1 & -1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & 1 & 1 & 1 & 1 & \\ & & & & 1 & 1 & 1 & -1 \\ & 0 & & & & 1 & 1 & 1 \\ & & & & & & 1 & 1 \end{bmatrix}$$

and define one type of lower Hessenberg matrix

$$\bar{H}_n = \begin{bmatrix} 2 & -1 & & & & & \\ 0 & 2 & 1 & & & & 0 \\ 1 & 0 & 2 & -1 & & & \\ & 1 & 0 & 2 & 1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & 0 & & 1 & 0 & 2 & (-1)^{n-1} \\ & & & & 1 & 0 & 1 \end{bmatrix}$$

For any natural number n. The study shows that permanents of these matrices can be realized as Pell numbers, Fibonacci numbers, Lucas numbers and their sums.

Similarly, in [2], Gulec studies an $n\times n$ tridiagonal matrix

$$H_n(s,t) = \begin{bmatrix} 2s & 2 & & & 0 \\ t & 2s & 1 & & \\ & t & 2s & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & t & 2s \end{bmatrix}$$

and proves that

$$perH_n(s,t) = perH_n^{n-2}(s,t) = q_n(s,t)$$

where $q_n(s,t)$ is the n th $(s,t)-$Pell Lucas number.

As a more intricated example, in [8], Minc considers a $n\times n(0,1)$ -matrix for which the i, j entry is 1 when $i-1\le j\le i+k-1$ and 0 otherwise for $k\le n+1$. Namely,

$$F(n,k)=\begin{bmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & 1 & 1 & \cdots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & \cdots & 0 & 1 & 1 & 1 \\ 0 & \cdots & \cdots & 0 & \cdots & 0 & 0 & 1 & 1 \end{bmatrix}.$$

His main result states that $per\,F(n,k) = g_{n+1}^k$, where g_n^k is the n th generalized order-*k* Fibonacci number, for $i=k$.

There are related researches on representing famous sequences as the permanent of tridiagonal matrices in [1,3,4,5,6,7]. In this article, we explore another type of tridiagonal matrices whose permanent result to the *k*-Jacobsthal sequence. We then show that the matrices are convertible.

2. MAIN RESULTS

Definition 1. Let $G_n=[g_{ij}]_{n\times n}$ be a tridiagonal matrix given by $g_{ii}=k,\ g_{i+1,i}=1,\ g_{i,i+1}=2$ for $1\le i\le n$ and 0 otherwise. Thus

$$G_n=\begin{bmatrix} k & 2 & & & 0 \\ 1 & k & 2 & & \\ & 1 & k & \ddots & \\ & & \ddots & \ddots & 2 \\ 0 & & & 1 & k \end{bmatrix}.$$

This article aims to prove that this given G_n has the permanent value in the *k*-Jacobsthal sequence.

Definition 2. For any natural number n, let S_n be the symmetric group on $\{1,2,3,...,n\}$ and $j=(j_1,j_2,...,j_n)$ be the element of this group. Then the permanent of an *n*-square matrix $A=[a_{ij}]_{n\times n}$ is

$$perA=\sum_{all j} a_{1j_1}a_{2j_2}...a_{nj_n},$$

where the summation extends over all permutations j in S_n.

Definition 3. Let $n\in\mathbb{N}$, $k>0$. Then, the *k*-Jacobsthal sequence $\{\hat{j}_{k,n}\}_{n\in\mathbb{N}}$ is defined by

$$\hat{j}_{k,n} = k\hat{j}_{k,n-1} + 2\hat{j}_{k,n-2},$$

with initial conditions $\hat{j}_{k,0}=0,\ \hat{j}_{k,1}=1$.

Namely, it is the sequence

$$0,1,\ k,\ k^2+2,\ k^3+4k,\ k^4+6k^2+4,\ k^5+8k^3+12k,\ ... .$$

Since the permanent of a matrix is as complex as the determinant, we will approach the problem by contractible matrices to gradually compute its value. For this reason, we define the following technical definitions below.

Definition 4. An $m \times n$ matrix $A=[a_{ij}]$ with row vectors $r_1, r_2, ..., r_m$ is contractible on column k if this column contains exactly two nonzero element.

Definition 5. If $A=[a_{ij}]$ is contractible on column k whose nonzero entries on column k are a_{ik}, a_{jk}, define the contraction of A on column k relative to rows i and j to be an $(m-1)\times(n-1)$ matrix $A_{ij:k}$ obtained from A replacing row i with a_{ik}, a_{jk} and deleting row j and column k is called the contraction of A on column k relative to row i and j.

Example 1.
Consider the matrix

$$A=\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 2 & 0 \\ 1 & -2 & 3 & 5 \\ 0 & 7 & -4 & 2 \end{bmatrix}.$$

Since $a_{22} \neq 0 \neq a_{32}$, A is contractible on column 2. Replacing row 2 with $a_{32}r_2 + a_{22}r_3$, we get

$$a_{21} = (-2)(3)+(4)(1) = -2$$
$$a_{22} = (-2)(4)+(4)(-2) = -16$$
$$a_{23} = (-2)(2)+(4)(3) = 8$$
$$a_{24} = (-2)(0)+(4)(5) = 20,$$

which transform the matrix to

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ -2 & -16 & 8 & 20 \\ 1 & -2 & 3 & 5 \\ 0 & 7 & -4 & 2 \end{bmatrix}.$$

After removing row 3 and column 2, we get

$$\begin{bmatrix} 1 & 0 & -1 \\ -2 & 8 & 20 \\ 0 & -4 & 2 \end{bmatrix}$$

as contraction of A on column 2 relative to row 2 and 3.

Proposition 1. Let A be a nonnegative integral matrix of order $n>1$. If B is a contraction of A then

$$perA = perB.$$

Theorem 1. Let n be a natural number and $\hat{j}_{k,n}$ be a k-Jacobsthal sequence. Then

$$perG_n = perG_n^{n-2} = \hat{j}_{k,n+1},$$

for which matrices G_n^r obtained by contracting G_n r times for any $1 \le r \le n-2$.

Proof We will prove by Mathematical induction. It is obvious when $n=1$ that $G_1=[k]$. Therefore,

$$perG_1 = k = \hat{j}_{k,2}.$$

For $n=2$, we see that $G_2 = \begin{bmatrix} k & 2 \\ 1 & k \end{bmatrix}$, so $perG_2 = k^2+2 = \hat{j}_{k,3}$.

Now for $n>2$, we use induction on l, the number of contractions performed to show that

$$G_n^l = \begin{bmatrix} j_{k,l+2} & 2j_{k,l+1} & & & 0 \\ 1 & k & 2 & & \\ & 1 & k & \ddots & \\ & & \ddots & \ddots & 2 \\ 0 & & & 1 & k \end{bmatrix},$$

for any $3 \le l \le n-4$.

For $l = 1$, since $g_{11}, g_{21} \ne 0$, we can contract G_n on column 1.

Firstly, r_1 is replaced by $g_{21}r_1 + g_{11}r_2$, and

$$\begin{aligned} g_{11} &= 1(k) + k(1) = 2k \\ g_{12} &= 1(2) + k(k) = k^2 + 2 \\ g_{13} &= 1(0) + k(2) = 2k \\ g_{14} &= 1(0) + k(0) = 0 \\ &\vdots \\ g_{1n} &= 1(0) + k(0) = 0. \end{aligned}$$

It follows immediately that

$$H = \begin{bmatrix} 2k & k^2+2 & 2k & & 0 \\ 1 & k & 2 & & \\ & 1 & k & \ddots & \\ & & \ddots & \ddots & 2 \\ 0 & & & 1 & k \end{bmatrix}.$$

Let us remove row 2 and column 1 from H to get the $(n-1)\times(n-1)$ contraction matrix G_n^1,

$$G_n^1 = \begin{bmatrix} k^2+2 & 2k & & & 0 \\ 1 & k & 2 & & \\ & 1 & k & \ddots & \\ & & \ddots & \ddots & 2 \\ 0 & & & 1 & k \end{bmatrix} = \begin{bmatrix} j_{k,3} & 2j_{k,2} & & & 0 \\ 1 & k & 2 & & \\ & 1 & k & \ddots & \\ & & \ddots & \ddots & 2 \\ 0 & & & 1 & k \end{bmatrix}.$$

Now, assume that

$$G_n^l = \begin{bmatrix} j_{k,l+2} & 2j_{k,l+1} & & & 0 \\ 1 & k & 2 & & \\ & 1 & k & \ddots & \\ & & \ddots & \ddots & 2 \\ 0 & & & 1 & k \end{bmatrix},$$

then, G_n^l is contractible on column 1, so we replace the first row by $g_{21}r_1 + g_{11}r_2$ to see that

$$\begin{aligned} a_{11} &= (1)\hat{j}_{k,l+2} + \hat{j}_{k,l+2}(1) \\ a_{12} &= (1)(2)\hat{j}_{k,l+1} + \hat{j}_{k,l+2}(k) \\ a_{13} &= (1)(0) + \hat{j}_{k,l+2}(2) \\ a_{14} &= (1)(0) + \hat{j}_{k,l+2}(0) \\ &\vdots \\ a_{1n} &= (1)(0) + \hat{j}_{k,l+2}(0). \end{aligned}$$

This results to the matrix

$$\bar{G} = \begin{bmatrix} 2\hat{j}_{k,l+2} & k\hat{j}_{k,l+2} + 2\hat{j}_{k,l+1} & 2\hat{j}_{k,l+2} & & 0 \\ 1 & k & 2 & & \\ & 1 & k & \ddots & \\ & & \ddots & \ddots & 2 \\ 0 & & & 1 & k \end{bmatrix}. \qquad (1)$$

By removing second row and first column of Equation 1, we obtain

$$G_n^{l+1} = \begin{bmatrix} \hat{j}_{k,(l+1)+2} & 2\hat{j}_{k,(l+1)+1} & & & 0 \\ 1 & k & 2 & & \\ & 1 & k & \ddots & \\ & & \ddots & \ddots & 2 \\ 0 & & & 1 & k \end{bmatrix},$$

which finish our induction.

We recall that $n > 2$. By choosing $l+1 = n-3$, one can achieve that

$$G_n^{n-3} = \begin{bmatrix} \hat{j}_{k,n-1} & 2\hat{j}_{k,n-2} & 0 \\ 1 & k & 2 \\ 0 & 1 & k \end{bmatrix}.$$

Lastly, we contract G_n^{n-3} on column 1 to compute the matrix

$$G_n^{n-2} = \begin{bmatrix} k\hat{j}_{k,n-1} + 2\hat{j}_{k,n-2} & 2\hat{j}_{k,n-1} \\ 1 & k \end{bmatrix} = \begin{bmatrix} \hat{j}_{k,n} & 2\hat{j}_{k,n-1} \\ 1 & k \end{bmatrix}.$$

Then finding the permanent

$$perG_n^{n-2} = k\hat{j}_{k,n} + 2\hat{j}_{k,n-1} = \hat{j}_{k,n+1}.$$

It follows immediately from Proposition 1 that

$$perG_n = perG_n^{n-2} = \hat{j}_{k,n+1}.$$

As an application of Theorem 1, we use is to prove an interesting property of G_n.

Definition 6. An $n \times n$ matrix A is convertible if there is an $(1,-1)$ matrix K such that

$$perA = det(A \circ K),$$

where $A \circ K$ denotes the Hadamard product of A and K.

Theorem 2. For any natural number n, G_n is convertible.

Proof. Let S_n be a $(1,-1)$ matrix of dimension $n \times n$ defined by

$$S_n = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & -1 & 1 \end{bmatrix}.$$

Then the Hadamard Product of G_n and S_n is

$$G_n \circ S_n = \begin{bmatrix} k(1) & 2(1) & 0(1) & 0(1) & \cdots & 0(1) \\ 1(-1) & k(1) & 2(1) & 0(1) & \cdots & 0(1) \\ 0(1) & 1(-1) & k(1) & 2(1) & \ddots & \vdots \\ 0(1) & 0(1) & 1(-1) & k(1) & \ddots & 0(1) \\ \vdots & \vdots & \ddots & \ddots & \ddots & 2(1) \\ 0(1) & 0(1) & \cdots & 0(1) & 1(-1) & k(1) \end{bmatrix} = \begin{bmatrix} k & 2 & & & 0 \\ -1 & k & 2 & & \\ & -1 & k & \ddots & \\ & & \ddots & \ddots & 2 \\ 0 & & & -1 & k \end{bmatrix}.$$

We use strong induction on n.
Base case: for $n = 1$, $G_1 \circ S_1 = [k]$. Hence $\det(G_1 \circ S_1) = k = j_{k,n}$.
Now assume that $\det(G_r \circ S_r) = j_{r,r+1}$ for any $1 < r \le k$.

Now consider

$$\begin{aligned} \det(G_{k+1} \circ S_{k+1}) &= a_{k+1,k+1} \det(G_k \circ S_k) - a_{k+1,k} a_{k,k+1} \det(G_{k-1} \circ S_{k-1}) \\ &= k j_{k,k+1} + 2 j_{k,k} \\ &= j_{k,k+2} \\ &= j_{k,(k+1)+1}. \end{aligned}$$

Therefore, $\det(G_n \circ S_n) = j_{k,n+1}$. By Theorem 1, we obtain

$$\det(G_n \circ S_n) = per G_n,$$

which finishes the proof.

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