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Application of Lie Algebras in Computer Animation

Atefeh Hasan-Zadeh

Fouman Faculty of Engineering, College of Engineering, University of Tehran
Fouman, Iran
Email: hasanzadeh.a [AT] ut.ac.ir

ABSTRACT— ***The mathematical theory behind the computer graphic enables one to develop the techniques for suitable creation of computer animation. This paper presents an application of Riemannian geometry in 3D animation via notions of in motion and deformation. By focusing on Lie algebras concepts, it provides a geometric framework for the implementation of computer animation.***

Keywords— computer graphic, 3D animation, geometric transformation, Lie algebra, covering map.

1. INTRODUCTION

Geometric transformations give a basic mathematical framework for geometric operations in computer graphics, such as rotation, shear, translation, and their compositions. Each affine transformation is then represented by a homogeneous matrix with usual operations: addition, scalar product, and product. While the product means the composition of the transformations, geometric meanings of addition and scalar product are not trivial. What is often considered is having a geometrically meaningful weighted sum (linear combination) of transformations, which is not an easy task. These kinds of practical demands, therefore, have inspired graphics researchers to explore new mathematical concepts and/or tools.

Many works have been conducted in this direction, including skinning [1-2], thinning [3], cage-based deformation [4], motion analysis and compression [5-7]. This paper focuses on a Lie theoretic aspect of the mathematical applications in computer graphics. For this purpose, Lie groups and their Lie algebras have been introduced. Then Lie algebras will correspond the associated Lie group of matrices as a motion group. As will be demonstrated, the Lie algebra gives a linear approximation of the Lie group, which allows one to use a powerful linear interpolation scheme in making dynamic motion and deformation.

After some preliminaries of differential geometry and Lie theory in Section 2, the application of the proposed theory has been developed to implementation of 3D animation.

2. BASIC CONCEPTS

According to common geometric notions (e.g., [8-10]) an n-dimensional manifold is a set M, together with countable coordinate charts $U_\alpha \subset M$ and one–to–one local coordinate maps $\phi_\alpha : U_\alpha \to V_\alpha$ onto connected open subsets $V_\alpha \subset \mathbf{R}^m$, which satisfy the following properties:

The coordinated charts cover M such as equation (1)

$$\bigcup_\alpha U_\alpha = M \tag{1}$$

and on the overlap of any pair of coordinate charts $U_\alpha \bigcap U_\beta$ the composite map (2)

$$\chi_\beta \circ \chi_\alpha^{-1} : \chi_\alpha\left(U_\alpha \bigcap U_\beta\right) \to \chi_\beta\left(U_\alpha \bigcap U_\beta\right) \tag{3}$$

is a smooth function. If $x \in U_\alpha$, $\tilde{x} \in U_\beta$ are distinct points of M, then, there exist open subsets $W \subset V_\alpha$, $\tilde{W} \subset V_\beta$, which satisfied with relations (3)

$$\chi_\alpha(x) \in W,\ \chi_\beta(\tilde{x}) \in \tilde{W},\ \chi_\alpha^{-1}(W) \cap \chi_\beta^{-1}(\tilde{W}) \neq \phi \tag{3}$$

A local r-parameter Lie group consists of open subsets $V_0 \subset V \subset \mathbf{R}^r$ containing origin, and the smooth maps $m : V \times V \to \mathbf{R}^r$ and $i : V_0 \to V$ defining the action group and the inversion action (respectively) which satisfy the property of associativity formulated in (4)

$$m(x, m(y,z)) = m(m(x,y),z),\ \forall x,y,z \in V \tag{4}$$

and identity as in relation (5)

$$m(0,x) = x = m(x,0),\ \forall x \in V \tag{5}$$

and inversion such as relation (6)

$$m(x, i(x)) = 0 = m(i(x), x),\ x \in V_0 \tag{6}$$

A local group of transformations acting on a manifold M is given by a (local) Lie group G, an open subset U as the domain of definition of the group action with $\{e\} \times M \subset U \subset G \times M$, and a smooth map $\Psi : U \to M$ which satisfy the properties formulated in (7)

$$(h,x) \in U,\ (g, \Psi(h,x)) \in U, (g.h, x) \in U \Rightarrow \Psi(g, \Psi(h,x)) = \Psi(g.h, x) \tag{7}$$

and the relation (8)

$$\Psi(e,x) = x,\ \forall x \in M \tag{8}$$

and the condition (9)

$$(g,x) \in U \Rightarrow (g, \Psi(h,x)) \in U,\ \Psi(g^{-1}, \Psi(g,x)) = x \tag{9}$$

For brevity, $\Psi(g,x)$ is shown as $g.x$.

At each point of a smooth parametrized curve (10)

$$\gamma : I \to M \tag{10}$$

of a subinterval of $\mathbf{R}$ on a manifold M , there is a tangent vector (11)

$$\dot{\gamma}(t) = \frac{d\gamma}{dt} = \left(\dot{\gamma}^1(t), \ldots, \dot{\gamma}^n(t)\right) \tag{11}$$

For an n-dimensional manifold M , $TM\big|_x$ which is a collection of all tangent vectors to all possible curves passing through a given point x in M forms the is an n-dimensional vector space, with the set as in (12)

$$\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m} \right\} \tag{12}$$

as a basis for it. A vector field v on M associate the tangent vector $v\big|_x \in TM\big|_x$ to any point $x \in M$ that $v\big|_x$ varies smoothly of each point to the other. In local coordinates $\left(x_1, \ldots, x_n\right)$, it is in the form (13)

$$v\big|_x = \xi^1(x)\frac{\partial}{\partial x_1} + \xi^2(x)\frac{\partial}{\partial x_2} + \ldots + \xi^n(x)\frac{\partial}{\partial x_n} \tag{13}$$

where each $\xi^i(x)$ is a smooth function of x . The maximal integral parametrized curves (14)

$$\dot{\gamma}(t) = v\big|_{\gamma(t)} \tag{14}$$

passing through $x \in M$ is shown by $\Psi(t, x)$, and is called the flow generated by a vector field v or a one-parameter group of transformations. In this case, the vector field v is called the infinitesimal generator of the action Ψ . Also the relation (15) is satisfy for this flow

$$\frac{d}{dt}\Psi(t, x) = v\big|_{\Psi(t,x)} \tag{15}$$

This flows is denoted by the relation (16)

$$\exp(tv)x \equiv \Psi(t, x) \tag{16}$$

which result in (17)

$$\frac{d}{dt}\left[\exp(tv)x\right] = v\big|_{\exp(tv)x},\ x \in M \tag{17}$$

There is a one-to-one correspondence between local one-parameter groups of transformations and their infinitesimal generators; for the vector field (18)

$$v = \sum \xi^i(x) \frac{\partial}{\partial x_i} \tag{18}$$

on M and the smooth function $f : M \to \mathbf{R}$, (19) is occluded

$$\frac{d}{dt} f(\exp(tv)x)\Big|_{t=0} = v(f)(x) \tag{19}$$

If v and w are vector fields on M, then their Lie bracket $[v,w]$ is the unique vector field satisfying (20)

$$[v,w](f) = v(w(f)) - w(v(f)) \tag{20}$$

for all smooth functions $f : M \to \mathbf{R}$. For any group element g of a Lie group G the right multiplication map $R_g : G \to G$ defined by (21)

$$R_g(h) = h.g \tag{21}$$

is a diffeomorphism, with inverse (22)

$$R_{g^{-1}} = (R_g)^{-1} \tag{22}$$

A vector field v on G is called right-invariant if the relation (23) is satisfied

$$dR_g(v|_h) = v|_{R_g(h)} = v|_{hg}, \ \forall h, g \in G \tag{23}$$

The set of all right-invariant vector fields forms a vector space. A Lie algebra is a vector space $\mathbf{G}$ with a bilinear operation $[.,.] : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$, called the Lie bracket for $\mathbf{G}$, satisfying the axioms (24)

$$\begin{aligned} [cv + c'v', w] &= c[v,w] + c'[v',w] \\ [v, +cw + c'w'] &= c[v,w] + c'[v,w'] \\ [v,w] &= -[w,v] \\ [u,[v,w]] + [w,[u,v]] + [v,[w,u]] &= 0 \end{aligned} \tag{24}$$

for all $c, c' \in \mathbf{R}$ and $u, v, w, v', w' \in G$. The flow generated by a right-invariant vector field $v \neq 0$ through the identity, namely (25)

$$g_t = \exp\left(tv\right)e \equiv \exp\left(tv\right) \tag{25}$$

is defined for all $t \in \mathbf{R}$ and forms a one-parameter subgroup of G. Conversely, any connected one-dimensional subgroup of G is generated by such a right-invariant vector field in the above manner.

3. RESULTS

Main Result. The details of creating a computer animation can be implemented in notions and differential geometry, especially, Lie algebras.

Proof. The axis of rotation is a natural invariant of $3D$ rotation. Let $B = \left\{x \in \mathbf{R}^3 \left\| x \right| \leq \pi\right\}$. Then the exponential map restricted to B gives a surjective map $\exp : B \to SO(3)$. It is diffeomorphic on the interior $\exp : \left\{x \in \mathbf{R}^3 \left\| x \right| \leq \pi\right\} \to: \left\{R \in SO(3) \left| \det(R+I) \neq \pi\right.\right\}$. On the boundary, it is a two-to-one covering map $\exp : \left\{x \in \mathbf{R}^3 \left\| x \right| = \pi\right\} \to: \left\{R \in SO(3) \left| \det(R+I) = \pi\right.\right\}$.

These two (rather distinct) behaviors are understood in a uniform manner: In a 3D figure, the map $\exp : \left\{x \in \mathbf{R}^3 \left| 0 < \left| x \right| \leq 2\pi\right.\right\} \to \left\{R \in SO(3) \left| R \neq I\right.\right\}$ gives the two-to-one covering map (everywhere smooth, so that the local inverse does exist uniquely). Slightly more generally, for every integer $n \geq 1$, $\exp : \left\{x \in \mathbf{R}^3 \left| 2(n-1)\pi < \left| x \right| \leq 2n\pi\right.\right\} \to \left\{R \in SO(3) \left| R \neq I\right.\right\}$ also gives the two-to-one covering map. This map factors through the map (26):

$$\exp : \left\{x \in \mathbf{R}^3 \left| 2(n-1)\pi < \left| x \right| \leq 2n\pi\right.\right\} \xrightarrow{\square} \left\{q \in S^3 \left| q \neq \pm 1\right.\right\} \xrightarrow{2:1} \left\{R \in SO(3) \left| R \neq I\right.\right\} \tag{26}$$

On the other hand, the exponential map on the complement is factored as the map (27):

$$\exp : \left\{x \in \mathbf{R}^3 \left\| x \right| = 2n\pi\right\} \to \left\{q \in S^3 \left| q \neq \pm 1\right.\right\} \to \left\{R \in SO(3)\right\} \tag{27}$$

where the first map is defined as: $x \mapsto (-1)^n$ for $\left| x \right| = 2n\pi$. The first map shows the degeneration of spheres $\exp : \left\{x \in \mathbf{R}^3 \left\| x \right| = 2n\pi\right\}$ which looks like circles in a 3D figure, to a point. By the degeneration (candy-wrapping operation), it can be obtained from tube-like body $\exp : \left\{x \in \mathbf{R}^3 \left| 2(n-1)\pi < \left| x \right| \leq 2n\pi\right.\right\}$. The second map of (26) collects the isomorphic S^3's for $n = 1, 2, \ldots$ into one piece. The left and right most points in the third stage are 1 and -1 in S^3, which were the joint points on the second stage. Then the third map of (26) applies it.

Also, this phenomenon can be understood by the following animation: consider the rotation around x-axis with 360 degrees and after that the rotation around y-axis with certain degree. It seems to be a continuous move, but there is not a continuous logarithmic lift of this motion.

After the first rotation, the transformation (matrix) remembers the axis of rotation, so that the sudden change of the rotation axis from x-axis to y-axis is considered to be a discontinuous move. Note that, if the move does not go through the identity, the continuous logarithmic lift always exists. If the move C^1 assume that a continuously differentiable (that is, the velocity is continuous), then the continuous logarithmic lift exists.

4. CONCLUSION

In this paper, a new geometric approach to structuring the details of creating a computer animation is presented. The results can be well implemented in the form of a computer program based on mathematical algorithms.

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