

General Formulations of Navier-Stokes Exact Solutions for Rotating Flow Systems with Variable Viscosity

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ABSTRACT— *Flows of variable viscosity fluids have many industrial applications in fluid mechanics and in engineering such as pump flow for high viscosity fluids. In most cases the fluid viscosity is mainly temperature dependent. Numerical investigation of such flows involves the solution of the Navier-Stokes equations with an extra difficulty arising from the fact that the viscosity is not constant over the flow field. This article presents an analytical solution of the Navier-Stokes equations for the case of laminar flows in rotating systems with variable viscosity fluids, aiming to provide reference solutions for the validation of numerical or empirical prediction models for such flows. In the present method, the analytical solution of the flow field is achieved by expressing the flow variables by using combination of Bessel and exponential functions. It is shown that the proposed solution satisfies the governing equations.*

Keywords— Navier-Stokes equations, exact solutions, variable viscosity, laminar, viscous flow

1. INTRODUCTION

In the previous years, problems of fluid flow through porous ducts have aroused the interest of Engineers and Mathematicians; the problems have been studied for their possible applications in cases of transpiration cooling, gaseous diffusion, drinking water treatment as well as biomedical engineering. The cases where an exact solution for the Navier-Stokes equations can be obtained are of particular importance in order to describe fluid motion of viscous flows. However, since the Navier-Stokes equations are non-linear, there cannot be a general method to solve analytically the full system of equations. Exact solutions on the other hand are very important for many reasons. They provide a reference solution to verify the accuracies of many approximate methods such as numerical and/or empirical ones. Although, nowadays, computer techniques make the complete integration of the Navier-Stokes equations feasible, the accuracy of numerical results can be established only by comparison with an exact solution [1]. Due to the non-linearity of the Navier-Stokes equations and the inapplicability of the superposition principle for non-linear partial differential equations, exact solutions are difficult to obtain. For this reason, only a limited number of exact solutions exist, which under certain assumptions a number of terms in the equations of motion either disappear automatically or may be neglected and the resulting equations reduce to a form that can be readily solved. Wang [2] has given an excellent review of these solutions of the Navier-Stokes equations.

A family of exact solutions was determined for steady plane motion of an incompressible fluid of variable viscosity with heat transfer. This method consists of flows for which the vorticity distribution is proportional to the stream function perturbed by an exponential stream. Defining a transformation variable, the governing Navier-Stokes equations are transformed into simple ordinary differential equations and a class of exact solution is obtained in [3].

The exact solutions of the Navier-Stokes equations when the viscosity is variable are rare, however the literature in which the viscosity is variable, is dependent upon the space, time, temperature, pressure etc. Martin [4] for the first time used an elegant method in the study of the Navier-Stokes equations for an incompressible fluid of variable viscosity. Martin reduced the order of the governing equations from second order to first order by introducing the vorticity function and the generalized energy function.

Naeem and Nadeem [5] generalized Martin’s approach to study the steady-state, plane, variable viscosity, solving the incompressible Navier-Stokes equations. They transformed the equations to a new system with viscosity, vorticity, speed and energy function. The transformation matrices included the unknown functions and determined some exact solutions for vortex, radial and parallel flows.

Naeem [6] presented recently a class of exact solutions of the equations governing the steady plane flows of incompressible fluid of variable viscosity for an originally specified vorticity distribution.

Some exact solutions of Navier-Stokes equations are also reported in the literature [7-13] in which the vorticity distribution is prescribed such that the governing equations written in terms of the stream function become linear.

Some researchers [14, 15] have used hodograph transformation in order to linearize the system of governing equations.

Some authors [16-18] have used inverse methods where some *a priori* conditions were assumed about the flow variables and have found some exact solutions. The above said solutions have been found for the flow of fluid with constant viscosity. But in many situations in the fluid flow, where the pressure and temperature gradients are high or in case of electrically conducting flow where the magnetic field plays dominant role, the viscosity is no longer constant.

The effects of linearly varying viscosity and thermal conductivity on steady free convective flow of a viscous incompressible fluid along an isothermal vertical plate in the presence of heat sink were investigated in [19]. The governing equations of continuity, momentum and energy are transformed into coupled and non-linear ordinary differential equations using similarity transformation and then solved using Runge-Kutta fourth order method.

The problem of heat transfer and entropy generation in the flow of a variable viscosity fluid passing through a cylindrical pipe with convective cooling was studied in [20].

The effect of variable viscosity together with thermal stratification on free convection flow of non-Newtonian fluids along a non-isothermal semi infinite horizontal plate embedded in a saturated porous medium was investigated in [21]. The governing equations of continuity, momentum and energy were transformed into non linear ordinary differential equations using similarity transformations and then solved by using Runge–Kutta–Gill method. Governing parameters for the problem under study were the variable viscosity, thermal stratification parameter, non – Newtonian parameter and the power law index parameter.

Variable viscosity Couette flow was investigated in [22] by solving analytically the Navier-Stokes equations using a perturbation method coupled with a Hermite-Padè approximation technique to obtain the velocity and temperature distributions.

In the present method, it is the first attempt, to the authors’ knowledge, that an analytical solution of the Navier-Stokes equations written in cylindrical coordinates is obtained for the case of incompressible flow for temperature dependent viscosity. It is proven that the analytical solution obtained satisfies the partial differential equations. This contribution aims to present a general solution of the equations. The specific boundary conditions depend from case to case and each researcher who wishes to apply the present method has just to implement the boundary conditions of his particular problem in order to obtain the appropriate solution.

2. GOVERNING EQUATIONS

Considering that the model aims to describe the motion of a Newtonian fluid, the Navier-Stokes equations are the governing equations of the problem [23]. It was chosen to express the equations in cylindrical coordinates because it is more convenient for axisymmetric bodies or rotating systems. Moreover since many applications of rotating systems concern fluid rotating machinery such as compressors, turbines or pumps, the relative frame of reference is preferred. In this case the relative velocity is linked to the absolute velocity and the rotation speed of the relative system of coordinates:

$$\vec{V} = \vec{W} + \vec{U} = \vec{W} + (\vec{\omega} \times \vec{r}) \tag{1}$$

where $\vec{V} = v_r \cdot \vec{i}_r + v_\theta \cdot \vec{i}_\theta + v_z \cdot \vec{i}_z$ is the absolute velocity vector, $\vec{W} = u_r \cdot \vec{i}_r + u_\theta \cdot \vec{i}_\theta + u_z \cdot \vec{i}_z$ is the relative velocity vector and $\vec{U} = (\vec{\omega} \times \vec{r}) \cdot \vec{i}_\theta$ is the rotating speed of the relative system of coordinates.

The continuity equation is [23]:

$$\frac{\partial \rho}{\partial t} = -(\nabla \cdot \rho v) \tag{2}$$

The momentum equation is [23]:

$$\frac{\partial(\rho v)}{\partial t} = -[\nabla \cdot \rho v v] - \nabla \cdot P - [\nabla \cdot \tau] + \rho \cdot g \quad (3)$$

The energy equation is [23]:

$$\frac{\partial(\rho E)}{\partial t} = -(\nabla \cdot \rho E v) - (\nabla \cdot q) - P \cdot (\nabla \cdot v) - (\tau : \nabla v) \quad (4)$$

The following simplified assumptions are made:

- steady state conditions, that means all partial derivatives of type $\partial(\dots)/\partial t$ are set to zero.
- incompressible flow, meaning that the fluid density is constant.
- circumferential variations of flow quantities are zero, that means all partial derivatives of type $\partial(\dots)/\partial \theta$ are set to zero.
- gravitational forces due to the fluid weight are negligible.

Having adopted the above simplifications, the partial differential equations take the form:

The continuity equation becomes:

$$\frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} = 0 \quad (5)$$

The system of the steady-state, incompressible Navier-Stokes equations can be written:

$$u_r \frac{\partial u_r}{\partial r} - \frac{u_\theta^2}{r} + u_z \frac{\partial u_r}{\partial z} = -\frac{1}{\rho} \cdot \frac{\partial P}{\partial r} + \frac{\mu}{\rho} \left[\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial z^2} \right] + \omega^2 \cdot r + 2 \cdot \omega \cdot u_\theta + \frac{1}{\rho} \cdot \frac{\partial \mu}{\partial z} \cdot \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) + \frac{2}{\rho} \cdot \frac{\partial \mu}{\partial r} \cdot \frac{\partial u_r}{\partial r} \quad (6)$$

$$u_r \frac{\partial u_\theta}{\partial r} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r \cdot u_\theta}{r} = -2 \cdot \omega \cdot u_r + \frac{\mu}{\rho} \left[\frac{\partial u_\theta^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} \right] + \frac{1}{\rho} \cdot \frac{\partial \mu}{\partial z} \cdot \frac{\partial u_\theta}{\partial z} + \frac{1}{\rho} \cdot \frac{\partial \mu}{\partial r} \cdot \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \quad (7)$$

$$u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho} \cdot \frac{\partial P}{\partial z} + \frac{\mu}{\rho} \left[\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial z^2} \right] + \frac{1}{\rho} \cdot \frac{\partial \mu}{\partial r} \cdot \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) + \frac{2}{\rho} \cdot \frac{\partial \mu}{\partial z} \cdot \frac{\partial u_z}{\partial z} \quad (8)$$

The energy equation is becoming:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \quad (9)$$

The following equation was chosen to express the viscosity in terms of temperature [23, 24]:

$$\mu = \mu_0 \cdot (1 - \beta \cdot T) \quad (10)$$

where β is the thermal expansion of the fluid.

3. NON-DIMENSIONALISATION

The system of the above partial differential equations can be written in non-dimensional form choosing the following parameters:

$$u^* = \frac{u_r}{\omega \cdot L}, v^* = \frac{u_\theta}{\omega \cdot L}, w^* = \frac{u_z}{\omega \cdot L}$$

$$z^* = \frac{z}{L}, R^* = \frac{r}{L}$$

where L is a characteristic length of the geometry in consideration.

$$P^* = \frac{P}{\rho \cdot (L \cdot \omega)^2}, T^* = \frac{c_p \cdot T}{(L \cdot \omega)^2}, H^* = \frac{\mu}{\mu_0}$$

$$Re = \frac{\rho \cdot L \cdot \omega^2}{\mu_0}, Ec = \frac{\beta \cdot (L \cdot \omega)^2}{c_p}$$

where Re and Ec are the non-dimensional Reynolds and Eckert numbers [23].

Using the above non-dimensional parameters, the continuity equation can be written:

$$\frac{\partial u^*}{\partial R^*} + \frac{u^*}{R^*} + \frac{\partial w^*}{\partial z^*} = 0 \quad (11)$$

Using the above non-dimensional parameters, the r -momentum equation can be written:

$$u^* \cdot \frac{\partial u^*}{\partial R^*} + w^* \cdot \frac{\partial u^*}{\partial z^*} - \frac{v^{*2}}{R^*} = -\frac{\partial P^*}{\partial R^*} + \frac{H^*}{Re} \left(\frac{\partial^2 u^*}{\partial R^{*2}} + \frac{1}{R^*} \cdot \frac{\partial u^*}{\partial R^*} - \frac{u^*}{R^*} + \frac{\partial^2 u^*}{\partial z^{*2}} \right) +$$

$$R^* + 2v^* + \frac{1}{Re} \cdot \frac{\partial H^*}{\partial z^*} \cdot \frac{\partial u^*}{\partial z^*} + \frac{1}{Re} \cdot \frac{\partial H^*}{\partial z^*} \cdot \frac{\partial w^*}{\partial R^*} + \frac{2}{Re} \cdot \frac{\partial H^*}{\partial R^*} \cdot \frac{\partial u^*}{\partial R^*} \quad (12)$$

The non-dimensional form of the θ -momentum equation is:

$$u^* \cdot \frac{\partial v^*}{\partial R^*} + w^* \cdot \frac{\partial v^*}{\partial z^*} + \frac{u^* \cdot v^*}{R^*} = -2 \cdot v^* + \frac{H^*}{Re} \left[\frac{\partial}{\partial R^*} \left(\frac{\partial v^*}{\partial R^*} \right) + \frac{1}{R^*} \cdot \frac{\partial v^*}{\partial R^*} - \frac{v^*}{R^{*2}} + \frac{\partial}{\partial z^*} \left(\frac{\partial v^*}{\partial z^*} \right) \right] +$$

$$+ \frac{1}{Re} \cdot \frac{\partial H^*}{\partial z^*} \cdot \frac{\partial v^*}{\partial z^*} + \frac{1}{Re} \cdot \frac{\partial H^*}{\partial R^*} \cdot \left(\frac{\partial v^*}{\partial R^*} - \frac{v^*}{R^*} \right) \quad (13)$$

The non-dimensional form of the z -momentum equation is:

$$u^* \cdot \frac{\partial w^*}{\partial R^*} + w^* \cdot \frac{\partial w^*}{\partial z^*} = -\frac{\partial P^*}{\partial z^*} + \frac{H^*}{Re} \left(\frac{\partial^2 w^*}{\partial R^{*2}} + \frac{1}{R^*} \cdot \frac{\partial w^*}{\partial R^*} + \frac{\partial^2 w^*}{\partial z^{*2}} \right) +$$

$$\frac{1}{Re} \cdot \frac{\partial H^*}{\partial R^*} \cdot \left(\frac{\partial u^*}{\partial z^*} + \frac{\partial w^*}{\partial R^*} \right) + \frac{2}{Re} \cdot \frac{\partial H^*}{\partial z^*} \cdot \frac{\partial w^*}{\partial z^*} \quad (14)$$

Using the above non-dimensional parameters, the energy equation is becoming:

$$\frac{\partial^2 T^*}{\partial R^{*2}} + \frac{1}{R^*} \cdot \frac{\partial T^*}{\partial R^*} + \frac{\partial^2 T^*}{\partial z^{*2}} = 0 \quad (15)$$

4. SOLUTION OF THE EQUATIONS

Resolving the system of equations (11) to (15), it was found that the axial velocity w^* , the radial velocity u^* , the tangential velocity v^* can be expressed in terms of the functions:

$$w^* = J_0(rb) \cdot e^{b \cdot z^*} \quad (16)$$

$$u^* = -J_1(rb) \cdot e^{b \cdot z^*} \quad (17)$$

$$v^* = -r \quad (18)$$

where $J_0(rb)$ and $J_1(rb)$ are the Bessel functions of the first kind and $b = 2,4056405$ defined in details in [25].

The pressure P^* , the temperature T^* and the viscosity H^* can be expressed in terms of the functions:

$$P^* = \frac{\xi}{2} \cdot [J_0^2(rb) + J_1^2(rb)] \cdot e^{2 \cdot b \cdot z^*} \quad (19)$$

$$\xi = \frac{2 \cdot b}{\text{Re}} - 1 \quad (20)$$

$$T^* = A + B \cdot J_0(rb) e^{bz} \quad (21)$$

$$H^* = J_0(b \cdot r) \cdot e^{b \cdot z} \quad (22)$$

where the constants $A = \frac{1}{Ec}$, $B = -\frac{1}{Ec}$

Implementing the above solutions (equations 16 to 22) to the non-dimensional partial differential equations (11) to (15) it is proven that the governing equations are satisfied.

4.1 Continuity equation

Substituting the solutions for u^* , w^* , v^* the continuity equation yields:

$$\frac{\partial \left(J_1 \cdot e^{-b \cdot z^*} e^{k \cdot t} + \frac{B}{R^*} \right)}{\partial R^*} + \frac{J_1 \cdot e^{-b \cdot z^*} e^{k \cdot t} + \frac{B}{R^*}}{R^*} + \frac{\partial \left(J_0 \cdot e^{-b \cdot z^*} e^{k \cdot t} + A \cdot (1 - R^{*2}) \right)}{\partial z^*} =$$

$$\frac{J_1(b \cdot R^*) \cdot e^{-b \cdot z^*} e^{k \cdot t}}{R^*} + \frac{B}{R^{*2}} + \left(b \cdot J_0(b \cdot R^*) - \frac{J_1(b \cdot R^*)}{R^*} \right) \cdot e^{-b \cdot z^*} e^{k \cdot t} + \left(-\frac{B}{R^{*2}} \right) + (-b) \cdot J_0(b \cdot R^*) \cdot e^{-b \cdot z^*} e^{k \cdot t} = 0$$

meaning that the continuity equation is satisfied.

4.2 R-momentum equation

Introducing the expressions for the flow velocities u^* , v^* , w^* , in the r -momentum equation, one can see that these expressions satisfy the equation.

The left hand side of the equation is:

$$\frac{\partial u^*}{\partial R^*} + w^* \cdot \frac{\partial u^*}{\partial z^*} - \frac{v^{*2}}{R^*} = b \cdot \left(e^{b \cdot z^*} \right)^2 \cdot J_0(b \cdot R^*) \cdot J_1(b \cdot R^*) - \frac{\left(e^{b \cdot z^*} \right)^2 \cdot \left(J_1(b \cdot R^*) \right)^2}{R^*} -$$

$$- b \cdot \left(e^{b \cdot z^*} \right)^2 \cdot J_0(b \cdot R^*) \cdot J_1(b \cdot R^*) = - \frac{e^{2 \cdot b \cdot z^*} \cdot \left(J_1(b \cdot R^*) \right)^2}{R^*} - R^*$$

The right hand-side terms of the equation are:

The term $\frac{\partial P^*}{\partial R^*}$ is becoming:

$$\begin{aligned} -\frac{\partial P^*}{\partial R^*} &= \frac{\left(\frac{2b}{\text{Re}} - 1\right) \cdot e^{2b \cdot z^*}}{2} \cdot \frac{\partial}{\partial R^*} \left[J_1^2(b \cdot R^*) + J_0^2(b \cdot R^*) \right] \\ &= \left(\frac{b}{\text{Re}} - \frac{1}{2}\right) \cdot e^{2b \cdot z^*} \cdot \left[2J_1(b \cdot R^*) \cdot \left(b \cdot J_0(b \cdot R^*) - \frac{J_1(b \cdot R^*)}{R^*} \right) + 2J_0(b \cdot R^*) \cdot \left(-b \cdot J_1(b \cdot R^*) \right) \right] = \\ &= \left(\frac{b}{\text{Re}} - \frac{1}{2}\right) \cdot e^{2b \cdot z^*} \cdot \frac{2J_1^2(b \cdot R^*)}{R^*} = \frac{2 \cdot b \cdot e^{2b \cdot z^*} \cdot J_1^2(b \cdot R^*)}{\text{Re} \cdot R^*} - \frac{e^{2b \cdot z^*} \cdot J_1^2(b \cdot R^*)}{R^*} \end{aligned}$$

The terms: $\frac{H^*}{\text{Re}} \cdot \left(\frac{\partial^2 u^*}{\partial R^{*2}} + \frac{1}{R^*} \cdot \frac{\partial u^*}{\partial R^*} - \frac{u^*}{R^*} + \frac{\partial^2 u^*}{\partial z^{*2}} \right) =$

$$\begin{aligned} &= \frac{H^*}{\text{Re}} \cdot \left[b^2 \cdot e^{b \cdot z^*} \cdot J_1(b \cdot R^*) + \frac{b \cdot e^{b \cdot z^*} \cdot J_0(b \cdot R^*)}{R^*} - \frac{2 \cdot e^{b \cdot z^*} \cdot J_1(b \cdot R^*)}{(R^*)^2} - \right. \\ &\quad \left. - \frac{b \cdot e^{b \cdot z^*} \cdot J_0(b \cdot R^*)}{R^*} + \frac{2 \cdot e^{b \cdot z^*} \cdot J_1(b \cdot R^*)}{(R^*)^2} - b^2 \cdot e^{b \cdot z^*} \cdot J_1(b \cdot R^*) \right] = 0 \end{aligned}$$

The terms: $R^* + 2v^* + \frac{1}{\text{Re}} \cdot \frac{\partial H^*}{\partial z^*} \cdot \frac{\partial u^*}{\partial z^*} + \frac{1}{\text{Re}} \cdot \frac{\partial H^*}{\partial z^*} \cdot \frac{\partial w^*}{\partial R^*} + \frac{2}{\text{Re}} \cdot \frac{\partial H^*}{\partial R^*} \cdot \frac{\partial u^*}{\partial R^*} =$

$$\begin{aligned} &= -R^* - \frac{2b^2 \cdot e^{2b \cdot z^*} \cdot J_0(b \cdot R^*) \cdot J_1(b \cdot R^*)}{\text{Re}} + \frac{2}{\text{Re}} \cdot \left(2b^2 \cdot e^{2b \cdot z^*} \cdot J_0(b \cdot R^*) \cdot J_1(b \cdot R^*) - \frac{2b \cdot e^{2b \cdot z^*} \cdot J_1^2(b \cdot R^*)}{R^*} \right) = \\ &= -R^* - \frac{2b \cdot e^{2b \cdot z^*} \cdot J_1^2(b \cdot R^*)}{\text{Re} \cdot R^*} \equiv \frac{\partial P^*}{\partial R^*} \end{aligned}$$

Thus, we see that the R-momentum equation satisfies the proposed solution.

4.3 Θ -momentum equation

Introducing the expressions for the flow velocities u^*, v^*, w^* , in the θ -momentum equation, one can see that these expressions satisfy the equation.

The left hand side of the equation is:

$$\begin{aligned} u^* \cdot \frac{\partial v^*}{\partial R^*} + w^* \cdot \frac{\partial v^*}{\partial z^*} + \frac{u^* \cdot v^*}{R^*} &= \left(-J_1(b \cdot R^*) \cdot e^{b \cdot z^*} \right) \cdot \frac{\partial(-R^*)}{\partial R^*} + \\ &\left(J_0(b \cdot R^*) \cdot e^{b \cdot z^*} \right) \cdot \frac{\partial(-R^*)}{\partial z^*} + \frac{\left(-J_1(b \cdot R^*) \cdot e^{b \cdot z^*} \right) \cdot (-R^*)}{R^*} = \\ -J_1(b \cdot R^*) \cdot e^{b \cdot z^*} + J_1(b \cdot R^*) \cdot e^{b \cdot z^*} &= 0 \end{aligned}$$

The right-hand side of the θ -momentum equation is:

The term

$$\frac{1}{\text{Re}} \cdot \frac{\partial H^*}{\partial z^*} \cdot \frac{\partial v^*}{\partial z^*} = \frac{1}{\text{Re}} \cdot \left(b \cdot J_0(b \cdot R^*) \cdot e^{b \cdot z^*} \right) \cdot \frac{\partial(-R^*)}{\partial z^*} = \frac{1}{\text{Re}} \cdot \left(b \cdot J_0(b \cdot R^*) \cdot e^{b \cdot z^*} \right) \cdot 0 = 0$$

The term

$$\frac{1}{\text{Re}} \cdot \frac{\partial H^*}{\partial R^*} \cdot \left(\frac{\partial v^*}{\partial R^*} - \frac{v^*}{R^*} \right) = \frac{1}{\text{Re}} \cdot \left(-b \cdot J_1(b \cdot R^*) \cdot e^{b \cdot z^*} \right) \cdot \left[\frac{\partial(-R^*)}{\partial R^*} - \frac{(-R^*)}{R^*} \right] = 0$$

The term

$$\frac{\omega^2 \cdot L \cdot H}{\text{Re}} \left[\frac{\partial}{\partial R^*} \left(\frac{\partial v^*}{\partial R^*} \right) + \frac{1}{R^*} \cdot \frac{\partial v^*}{\partial R^*} - \frac{v^*}{R^{*2}} + \frac{\partial}{\partial z^*} \left(\frac{\partial v^*}{\partial z^*} \right) \right] =$$

$$\frac{J_0(b \cdot R^*) \cdot e^{b \cdot z^*}}{\text{Re}} \cdot \left[\frac{\partial}{\partial R^*} (-1) + \frac{1}{R^*} \cdot (-1) + \frac{R^*}{R^{*2}} + 0 \right] = \frac{J_0(b \cdot R^*) \cdot e^{b \cdot z^*}}{\text{Re}} \cdot \left(-\frac{1}{R^*} + \frac{1}{R^*} \right) = 0$$

Hence the proposed solution satisfies the θ -momentum equation.

4.4 Z-momentum equation

Introducing the expressions for the flow velocities u^*, v^*, w^* , in the z -momentum equation, one can see that these expressions satisfy the equation.

The left hand side of the equation is:

$$u^* \cdot \frac{\partial w^*}{\partial R^*} + w^* \cdot \frac{\partial w^*}{\partial z^*} =$$

$$= \left(-J_1(b \cdot R^*) \cdot e^{b \cdot z^*} \right) \cdot \left(-b \cdot e^{b \cdot z^*} \cdot J_1(b \cdot R^*) \right) + \left(J_0(b \cdot R^*) \cdot e^{b \cdot z^*} \right) \cdot \frac{\partial \left(J_0(b \cdot R^*) \cdot e^{b \cdot z^*} \right)}{\partial z^*}$$

$$= b \cdot e^{2 \cdot b \cdot z^*} \cdot \left[J_1^2(b \cdot R^*) + J_0^2(b \cdot R^*) \right]$$

The term $-\frac{\partial P^*}{\partial z^*}$ can be written:

$$-\frac{\partial P^*}{\partial z^*} = -\frac{\partial \left[\frac{\xi}{2} \cdot \left(J_1^2(b \cdot R^*) + J_0^2(b \cdot R^*) \right) \right]}{\partial z^*} = -\frac{\xi}{2} \cdot \left[J_1^2(b \cdot R^*) + J_0^2(b \cdot R^*) \right] \cdot \frac{\partial \left(e^{2 \cdot b \cdot z^*} \right)}{\partial z^*} =$$

$$-\xi \cdot \left[J_1^2(b \cdot R^*) + J_0^2(b \cdot R^*) \right] \cdot b \cdot e^{2 \cdot b \cdot z^*} = b \cdot e^{2 \cdot b \cdot z^*} \cdot \left(\frac{2 \cdot b}{\text{Re}} - 1 \right) \cdot \left[J_1^2(b \cdot R^*) + J_0^2(b \cdot R^*) \right]$$

The term $\frac{H^*}{\text{Re}} \cdot \left(\frac{\partial^2 w^*}{\partial R^{*2}} + \frac{1}{R^*} \cdot \frac{\partial w^*}{\partial R^*} + \frac{\partial^2 w^*}{\partial z^{*2}} \right)$ can be written:

$$\frac{H^*}{\text{Re}} \cdot \left(\frac{\partial^2 w^*}{\partial R^{*2}} + \frac{1}{R^*} \cdot \frac{\partial w^*}{\partial R^*} + \frac{\partial^2 w^*}{\partial z^{*2}} \right) =$$

$$\frac{J_0(b \cdot R^*) \cdot e^{b \cdot z^*}}{\text{Re}} \cdot \left[-b^2 \cdot e^{b \cdot z^*} \cdot J_0(b \cdot R^*) + b \cdot e^{b \cdot z^*} \cdot \frac{J_1(b \cdot R^*)}{R^*} - b \cdot e^{b \cdot z^*} \cdot \frac{J_1(b \cdot R^*)}{R^*} + b^2 \cdot e^{b \cdot z^*} \cdot J_0(b \cdot R^*) \right] = 0$$

The term $\frac{1}{\text{Re}} \cdot \frac{\partial H^*}{\partial R^*} \cdot \left(\frac{\partial u^*}{\partial z^*} + \frac{\partial w^*}{\partial R^*} \right)$ can be written as:

$$\frac{1}{\text{Re}} \cdot \frac{\partial H^*}{\partial R^*} \cdot \left(\frac{\partial u^*}{\partial z^*} + \frac{\partial w^*}{\partial R^*} \right) = \frac{2 \cdot b^2 \cdot e^{2 \cdot b \cdot z^*} \cdot J_1^2(b \cdot R^*)}{\text{Re}}$$

The term $\frac{2}{\text{Re}} \cdot \frac{\partial H^*}{\partial z^*} \cdot \frac{\partial w^*}{\partial z^*}$ can be written as:

$$\frac{2}{\text{Re}} \cdot \frac{\partial H^*}{\partial z^*} \cdot \frac{\partial w^*}{\partial z^*} = \frac{2}{\text{Re}} \cdot \left[\left(b \cdot e^{b \cdot z^*} \cdot J_0(b \cdot R^*) \right) \cdot \left(b \cdot e^{b \cdot z^*} \cdot J_0(b \cdot R^*) \right) \right] = \frac{2 \cdot b^2 \cdot e^{2 \cdot b \cdot z^*} \cdot J_0^2(b \cdot R^*)}{\text{Re}}$$

Substituting the above expressions to the non-dimensional form of the z-momentum equation, we obtain:

$$b \cdot e^{2 \cdot b \cdot z^*} \cdot \left[J_1^2(b \cdot R^*) + J_0^2(b \cdot R^*) \right] = -b \cdot e^{2 \cdot b \cdot z^*} \cdot \left(\frac{2 \cdot b}{\text{Re}} - 1 \right) \cdot \left[J_1^2(b \cdot R^*) + J_0^2(b \cdot R^*) \right] \\ + \frac{2 \cdot b^2 \cdot e^{2 \cdot b \cdot z^*} \cdot J_1^2(b \cdot R^*)}{\text{Re}} + \frac{2 \cdot b^2 \cdot e^{2 \cdot b \cdot z^*} \cdot J_0^2(b \cdot R^*)}{\text{Re}}$$

which means that proposed solution for the temperature satisfies the z-momentum equation.

4.5 Energy equation

$$\frac{\partial^2 T^*}{\partial R^{*2}} + \frac{1}{R^*} \cdot \frac{\partial T^*}{\partial R^*} + \frac{\partial^2 T^*}{\partial z^{*2}} = 0$$

Since the proposed solution for temperature is:

$$T^* = A + B \cdot J_0(b \cdot R^*) \cdot e^{b \cdot z^*}, \text{ the term } \frac{\partial T^*}{\partial R^*} \text{ can be expressed as:}$$

$$\frac{\partial T^*}{\partial R^*} = \frac{\partial \left(A + B \cdot J_0(b \cdot R^*) \cdot e^{b \cdot z^*} \right)}{\partial R^*} = -B \cdot b \cdot e^{b \cdot z^*} \cdot J_1(b \cdot R^*)$$

The term $\frac{\partial T^*}{\partial z^*}$ can be expressed as:

$$\frac{\partial T^*}{\partial z^*} = \frac{\partial \left(A + B \cdot J_0(b \cdot R^*) \cdot e^{b \cdot z^*} \right)}{\partial z^*} = B \cdot b \cdot e^{b \cdot z^*} \cdot J_0(b \cdot R^*)$$

The derivative $\frac{\partial \left(-B \cdot b \cdot e^{b \cdot z^*} \cdot J_1(b \cdot R^*) \right)}{\partial R^*}$ is expressed as:

$$\frac{\partial \left(-B \cdot b \cdot e^{b \cdot z^*} \cdot J_1(b \cdot R^*) \right)}{\partial R^*} = -b^2 \cdot B \cdot e^{b \cdot z^*} \cdot J_0(b \cdot R^*) + \frac{B \cdot b \cdot e^{b \cdot z^*} \cdot J_1(b \cdot R^*)}{R^*}$$

The derivative $\frac{\partial \left(B \cdot b \cdot e^{b \cdot z^*} \cdot J_0(b \cdot R^*) \right)}{\partial z^*}$ is expressed as:

$$\frac{\partial \left(B \cdot b \cdot e^{b \cdot z^*} \cdot J_0(b \cdot R^*) \right)}{\partial z^*} = b^2 \cdot B \cdot e^{b \cdot z^*} \cdot J_0(b \cdot R^*)$$

Thus the energy equation is:

$$-b^2 \cdot B \cdot e^{b \cdot z^*} \cdot J_0(b \cdot R^*) + \frac{B \cdot b \cdot e^{b \cdot z^*} \cdot J_1(b \cdot R^*)}{R^*} - \frac{B \cdot b \cdot e^{b \cdot z^*} \cdot J_1(b \cdot R^*)}{R^*}$$

$$+b^2 \cdot B \cdot e^{b \cdot z^*} \cdot J_0(b \cdot R^*) = 0$$

which means that the proposed solution for the temperature satisfies the energy equation.

5. CONCLUSIONS

In this article, an original work has presented an exact solution of the Navier-Stokes equations in cylindrical coordinates for incompressible, laminar axisymmetric variable viscosity flows in rotating systems. The fluid viscosity was assumed to be a function of temperature. The solution field consists of the Bessel functions of the first kind and of exponential functions. It was shown that equations (16) to (22) form a solution of the system of the Navier Stokes equations governing the flow field. Thus the present method can be used to provide reference solutions for numerical and empirical methods for flow field predictions in rotating systems involving fluids of variable viscosity.

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