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Application of Differential Transform Method to Integral and Integro-Differential Equations

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ABSTRACT--- -*In this paper, Differential Transform Method (DTM) has been used to solve some systems of linear and nonlinear Integro-differential equations. The approximate solution in the form of a series are calculated with easily computable terms. The solution obtained using this method is compared with the solution obtained using existing methods.*

Keywords---- Initial conditions**,** Integro-differential equations, Volterra equations, Linear and nonlinear system, Differential Transform Method, Analytic solution

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1. INTRODUCTION

The Differential Transform Method (DTM) is a method for solving a wide range of problems whose mathematical models yield equations or systems of equations involving algebraic, differential,integral and integro-differential equations. The concept of the differential transform was first proposed by Zhou (1986) where in both linear and non-linear initial value problems in electric circuit analysis [1] were solved. This method constructs an analytical solution in the form of polynomials. It is different from the high-order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally expansive for large orders. The differential transform is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. In recent years the application of differential transform theory has been appeared in many researches.

The theory and application of integro-differential equation is an important subject with in applied mathematics. A large class of scientific and engineering problems modelled by partial differential equations can be expressed in various forms of differential or integro-differential equations in abstract spaces. Integro-differential equations include many physical phenomena such as heat flow in materials with memory, viscoelasticity, heat conduction and wave propagation. Quasilinear integro-differential equation is also a factor which describes the study of nonlinear behavior of elastic strings and nonlinear conservative law with memory. One of the most important fields of modern research is the distributed control systems which is exercised through the boundary in a different way[2].

These are motivations to solve these kind of equations. Using the DTM several examples of linear and nonlinear integro-differential equations are tested and the results reveal that the DTM is very effective and simple.

In this paper,two linear and two nonlinear integro-differential equations are solved. In Section 2, some properties of DTM are given. In Section 3 DTM has been applied to solve linear and nonlinear integro-differential equations.

2. DIFFERENTIAL TRANSFORM METHOD

In this section, some basic properties of differential transform method are given. The differential transform of a function $f(y)$ is defined as follows.

$$F(k) = \frac{1}{k!}\left[\frac{\mathrm{d}^k f(y)}{\mathrm{d}y^k}\right], \tag{2.1}$$

where $f(y)$ is the original function and $F(k)$ is the differential transform of $f(y)$.

The differential inverse transforms of $F(k)$ is defined as

$$f(y) = \sum_{k=0}^{\infty} F(k)(y-y_0)^k. \tag{2.2}$$

From (2.1) and (2.2) we get

$$f(y) = \sum_{k=0}^{\infty} \frac{(y-y_0)^k}{k!}\frac{\mathrm{d}^k f(y)}{\mathrm{d}y^k}, \tag{2.3}$$

which implies that the concept of differential transform is derived from Taylor series expansion,but the method doesnot evaluate the derivatives symbolically. In real applications, the function $f(y)$ is expressed by a finite series and the equation (2.2) can be written as

$$f(y)=\sum_{k=0}^{n} F(k)(y-y_0)^k. \qquad (2.4)$$

From equations (2.1) and (2.2) we obtain the fundamental operations of DTM.

If $u(y)$ and $v(y)$are two functions where $U(k)$and $V(k)$ are the differential transforms corresponding to $u(y)$ and $v(y)$ then the following properties hold good [3, 4]:

1. If $f(y)=u(y)\pm v(y)$, then $F(k)=U(k)\pm V(k)$.
2. If $f(y)=\alpha u(y)$, then $F(k)=\alpha U(k)$.
3. If $f(y)=u(y)v(y)$, then $F(k)=\sum_{l=0}^{k} U(l)V(k-l)$.
4. If $f(y)=\frac{\mathrm{d}u(y)}{\mathrm{d}x}$, then $F(k)=(k+1)U(k+1)$.
5. If $f(y)=\frac{\mathrm{d}^m u(y)}{\mathrm{d}x^m}$, then $F(k)=(k+1)\ldots\ldots(k+m)U(k+m)$.
6. If $f(y)=\int_{y_0}^{y} u(t)\mathrm{d}t$, then $F(k)=\frac{U(k-1)}{k}$, $k\geq 1$, $F(0)=0$.
7. If $f(y)=y,^m$ then $F(k)=\delta(k-m)$.

In the following theorem we find the differential transformation for the product of two single-valued functions. This result is very useful on our approach for solving integral equations.

Theorem: *Suppose that $U(k)$ and $V(k)$ are the differential transformations of the functions $u(y)$ and $v(y)$ respectively then we have the following property. If $f(y)=\int_{y_0}^{y} v(t)u(t)dt$ then*

$$F(k)=\sum_{l=0}^{k-l}\frac{V(l)U(k-l-1)}{k},\ F(0)=0. \qquad (2.5)$$

3. APPLICATION OF THE DTM

In order to illustrate the advantages and the accuracy of the DTM for solving the linear and nonlinear integral and integro-differential equations we have applied the method to integro-differential equations.

Example 3.1. Consider the linear system of Volterra integro-differential equation [5]

$$u'(x)=1+x+x^2-v(x)-\int_0^x[u(t)+v(t)]\mathrm{d}t \qquad (3.1)$$
$$v'(x)=-1-x+u(x)-\int_0^x[u(t)-v(t)]\mathrm{d}t \qquad (3.2)$$

with initial conditions $u(0)=1,\quad v(0)=-1.$ (3.3)

Taking the differential transformation of equations(3.1) and(3.2), one gets

$$U(k+1)=\frac{1}{k+1}\left[\delta(k)+\delta(k-1)+\delta(k-2)-V(k)-\frac{U(k-1)}{k}-\frac{V(k-1)}{k}\right],\quad k\geq 1, \qquad (3.4)$$
$$V(k+1)=\frac{1}{k+1}\left[-\delta(k)-\delta(k-1)+U(k)-\frac{U(k-1)}{k}+\frac{V(k-1)}{k}\right],\quad k\geq 1. \qquad (3.5)$$

Taking the differential transformation of equation (3.3) leads to

$$U(0)=1,\ V(0)=-1. \qquad (3.6)$$

Putting $x=0$ in (3.1) and (3.2) we get $u'(0)=2, v'(0)=0$ and hence $U(1)=2$ and $V(1)=0$.
For $k=1,2,3,\ldots$,the series coefficients for U and V obtained from (3.4) and(3.5),*viz.*

$$U(2)=\frac{1}{2!};\quad U(3)=\frac{1}{3!};U(4)=\frac{1}{4!};\quad U(5)=\frac{1}{5!} \qquad (3.7)$$

And so on and

$$V(2)=-\frac{1}{2!};\quad V(3)=-\frac{1}{3!};\quad V(4)=-\frac{1}{4!};\quad V(5)=-\frac{1}{5!} \qquad (3.8)$$

and so on.
Then the successive approximations to the solution are obtained as

$$u_0=1,\quad u_1=1+2x,\quad u_2=1+2x+\frac{1}{2!}x^2,\quad u_3=1+2x+\frac{1}{2!}x^2+\frac{1}{3!}x^3,$$
$$u_4=1+2x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4,\quad u_5=1+2x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5,\ldots$$

and

$$v_0=-1,\quad v_1=0, v_2=-1-\frac{1}{2!}x^2,\quad v_3=-1-\frac{1}{2!}x^2-\frac{1}{3!}x^3,$$
$$v_4=-1-\frac{1}{2!}x^2-\frac{1}{3!}x^3-\frac{1}{4!}x^4,\quad v_5=-1-\frac{1}{2!}x^2-\frac{1}{3!}x^3-\frac{1}{4!}x^4-\frac{1}{5!}x^5,\ldots$$

We see that $u_n(x)=\sum_{k=0}^{n} U(k)x^k$converges to the exact solution $x+\mathrm{e}^x$ and that $v_n(x)=\sum_{k=0}^{n} V(k)x^k$converges to the exact solution $x-\mathrm{e}^x$.

Example3.2. Consider the system of nonlinear integro-differential equation

$$u''(x)=1-\frac{1}{3}x^3-\frac{1}{2}v'^2(x)+\frac{1}{2}\int_0^x[u^2(t)+v^2(t)]\mathrm{d}t \qquad (3.9)$$

$v''(x) = -1 + x^2 - xu(x) + \frac{1}{4}\int_0^x [u^2(t) - v^2(t)]\mathrm{d}t$ (3.10)

with the initial conditions

$u(0) = 1, \quad u'(0) = 2, \quad v(0) = -1, \quad v'(0) = 0.$ (3.11)

Taking the differential transformation of equations (3.9), (3.10), and (3.11), one gets

$$U(k+2) = \frac{1}{(k+1)(k+2)}\left\{\delta(k) - \frac{1}{3}\delta(k-3) - \frac{1}{2}\{\sum_{r=0}^{k}(r+1)(k-r+1)V(r+1)V(k-r+1)\} + \frac{1}{2}\left[\sum_{l=0}^{k-1}\frac{U(l)U(k-l-1)}{k}\right] + \frac{1}{2}\left[\sum_{m=0}^{k-1}\frac{V(m)V(k-m-1)}{k}\right]\right\}, \quad k \geq 1, \quad (3.12)$$

$$V(k+2) = \frac{1}{(k+1)(k+2)}\left\{-\delta(k) + \delta(k-2) - U(k-1) + \frac{1}{4}\left[\sum_{l=0}^{k-1}\frac{U(l)U(k-l-1)}{k}\right] - \frac{1}{4}\left[\sum_{m=0}^{k-1}\frac{V(m)V(k-m-1)}{k}\right]\right\}, \quad k \geq 1. \quad (3.13)$$

$U(0) = 1, \quad U(1) = 2, \quad V(0) = -1, \quad V(1) = 0.$ (3.14)

Putting $x = 0$ in (3.9) and (3.10) we get $u''(0) = 1$, $v''(0) = -1$ and hence $U(2) = \frac{1}{2!}$ $V(2) = -\frac{1}{2!}$.
For $k = 1, 2, 3, \ldots$, the series coefficients for U and V obtained from (3.12) and(3.13). The convergence of the sequences u_n and v_n is the same as in Example 3.1 since this problem also has the same exact solution $u(x) = x + \mathrm{e}^x$and $v(x) = x - \mathrm{e}^x$.

*Example 3.3.*Consider the integro-differential equation with polynomial non-linearity [5, 6]

$u'(x) = -1 + \int_0^x u^2(t)\mathrm{d}t$ (3.15)

for$x \in [0,1]$ with the condition $u(0) = 0$. (3.16)

Taking the differential transform of equation (3.15) and (3.16)one gets

$U(k+1) = -\delta(k) + \frac{1}{k+1}\left[\sum_{l=0}^{k-1}\frac{U(l)U(k-l-1)}{k}\right], \quad k \geq 1.$ (3.17)

$U(0) = 0.$ (3.18)

Putting $x = 0$ in (3.15), we get $u'(0) = -1$and hence $U(1) = -1$.
For k =1,2,3, . . . ,the series coefficients for Uobtained from (3.16), *viz.*,

$U(2) = 0; \quad U(3) = 0; \quad U(4) = \frac{1}{12}; U(5) = 0; \quad U(6) = 0; \quad U(7) = -\frac{1}{252}, \ldots$ (3.19)

and so on.
Then the successive approximations to the solution are obtained as

$$u_0 = 0, \quad u_1 = -x = u_2 = u_3, \quad u_4 = -x + \frac{1}{12}x^4 = u_5 = u_6, \quad u_7 = -x + \frac{1}{12}x^4 - \frac{1}{252}x^7, \ldots$$

Table 1 gives the values of $u_n(x)$evaluated at $x =$ 0.0000, 0.0625, 0.1250, 0.1875, 0.2500, 0.3125, 0.3750, 0.4375, 0.5000,...0.9375,1.0000 for different values of n.

Table1. Convergence of sequenceu_n

x	$u_1 = u_2 = u_3$	$u_4 = u_5 = u_6$	u_7
0.00000	0.00000	0.00000	0.00000
0.0625	−0.0625	−0.06249	−0.06249
0.1250	−0.1250	−0.12498	−0.12498
0.1875	−0.1875	−0.18739	−0.18739
0.2500	−0.2500	−0.24967	−0.24968
0.3125	−0.3125	−0.31171	−0.31171
0.3750	−0.3750	−0.37335	−0.37336
0.4375	−0.4375	−0.43445	−0.43446
0.5000	−0.5000	−0.49479	−0.49482
0.5625	−0.5625	−0.55416	−0.55423
0.6250	−0.6250	−0.61228	−0.61243
0.6875	−0.6875	−0.66888	−0.66917
0.7500	−0.7500	−0.72363	−0.72416
0.8125	−0.8125	−0.77618	−0.77711
0.8750	−0.8750	−0.82615	−0.82771
0.9375	−0.9375	−0.87313	−0.87565
1.0000	−1.0000	−0.91667	−0.92063

Table 2 contains a numerical comparison between the solution obtained using Differential Transform Method (DTM), and the solution of the same problem presented in [6] using LADA, [7] using ADM and [8] using VIM.

Table2. Comparison with the existing results

x	Present Method[u_7]	LADA [6]	ADM [7]	VIM [8]
0.00000	0.000000	0.000000	0.000000	0.000000
0.06250	−0.062498	−0.062498	−0.062498	−0.062698
0.12500	−0.124980	−0.124979	−0.124980	−0.124979
0.18750	−0.187390	−0.187397	− 0.187397	−0.187397
0.25000	−0.249675	−0.249674	−0.249674	−0.249674
0.31250	−0.311710	−0.311706	−0.311706	−0.311706
0.37500	−0.373356	−0.373356	−0.373356	−0.373356
0.43750	−0.434459	−0.434459	−0.434459	−0.434459
0.50000	−0.494823	−0.494823	−0.494823	−0.494823
0.56250	−0.554227	−0.554227	−0.554227	−0.554228
0.62500	−0.612431	−0.612431	−0.612431	−0.612432
0.68750	−0.669167	−0.669167	−0.669167	−0.669172
0.75000	−0.724163	−0.724153	−0.724153	−0.724166
0.81250	−0.777110	−0.777090	−0.777091	−0.777120
0.87500	−0.827709	−0.827667	−0.827667	−0.827731
0.93750	−0.875653	−0.875569	−0.875569	−0.875695
1.00000	−0.920634	−0.920476	−0.920476	−0.920717

In DTM only up to the 7th power is taken whereas the results of LADA used up to 19th power, and in ADM up to 13th power were used. Also it can be seen that VIM slightly over predicts the values, while the other three methods coincide with each other.

*Example3.4.*Consider the system of linear Volterra integro-differential equations [9]

$$u''(x) = -1 - x^2 - \sin x + \int_0^x [u(t) + v(t)]dt \quad (3.20)$$

$$v''(x) = 1 - 2\sin x - \cos x + \int_0^x [u(t) - v(t)]dt \quad (3.21)$$

with the initial conditions

$$u(0) = 1,\ u'(0) = 1,\ v(0) = 0,\ v'(0) = 2. \quad (3.22)$$

Taking the differential transform of equation (3.20), (3.21), and (3.22)one gets

$$U(k+2) = \frac{1}{(k+1)(k+2)}\left[-\delta(k-0) - \delta(k-2) - \frac{\sin\left(\frac{\pi k}{2}\right)}{k!} + \frac{U(k-1)}{k} + \frac{V(k-1)}{k}\right], \quad k \geq 1, \quad (3.23)$$

$$V(k+2) = \frac{1}{(k+1)(k+2)}\left[\delta(k) - \frac{2\sin(\pi k/2)}{k!} - \frac{\cos(\pi k/2)}{k!} + \frac{U(k-1)}{k} - \frac{V(k-1)}{k}\right], \quad k \geq 1, \quad (3.24)$$

$$U(0) = 1,\quad U(1) = 1,\quad V(0) = 0,\quad V(1) = 2. \quad (3.25)$$

Putting $x = 0$ in (3.20) and (3.21),we get $u''(0) = -1,\ v''(0) = 0$ and hence $U(2) = -\frac{1}{2}$, $V(2) = 0$.

For k =1, 2, 3,. . .,the series coefficients for U and V obtained from (3.23) and(3.24),as

$$U(3) = 0; U(4) = \frac{1}{2.3.4};\ U(5) = 0; \quad U(6) = -\frac{1}{2.3.4.5.6}; \ldots \quad (3.26)$$

and so on and

$$V(3) = -\frac{1}{2.3}; \quad V(4) = 0; \quad V(5) = \frac{1}{2.3.4.5}; \quad V(6) = 0; \ldots \quad (3.27)$$

and so on.

Then the successive approximations to the solution are obtained as

$u_0(x) = 1, u_1(x) = 1 + x, \quad u_2(x) = 1 + x - \frac{1}{2}x^2 = u_3(x),$

$u_4(x) = 1 + x - \frac{1}{2}x^2 + \frac{1}{4!}x^4 = u_5(x), \quad u_6(x) = 1 + x - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6, \ldots$

and

$v_0(x) = 0, \quad v_1(x) = 2x = v_2(x), \quad v_3(x) = 2x - \frac{1}{3!}x^3 = v_4(x), \quad v_5(x) = 2x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5, \ldots$

We see that $u_n(x) = \sum_{k=0}^{n} U(k)x^k$converges to the exact solution $x + \cos x$ and $v_n(x) = \sum_{k=0}^{n} V(k)x^k$ converges to the exact solution $x + \sin x$.

4. CONCLUSION

In this work differential transform method has been used successfully for solving the systems of integral equations.The results reveal the efficiency of this method for solving these systems. The present method reduces the computational difficulties of the other usual methods and the calculations are done easily.

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