On Exact Solutions of Phi-4 Partial Differential Equation Using the Enhanced Modified Simple Equation Method

Ziad Salem Rached

Department of Mathematics, Notre Dame University-Louaize
Zouk Mosbeh, Lebanon
Email: zrached [AT] ndu.edu.lb

ABSTRACT---- ***Constructing exact solutions of nonlinear ordinary and partial differential equations is an important topic in various disciplines such as Mathematics, Physics, Engineering, Biology, Astronomy, Chemistry,… since many problems and experiments can be modeled using these equations. Various methods are available in the literature to obtain explicit exact solutions. In this correspondence, the enhanced modified simple equation method (EMSEM) is applied to the Phi-4 partial differential equation. New exact solutions are obtained.***

Keywords--- Nonlinear differential equation, exact solutions, EMSEM, Phi-4 equation

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1. INTRODUCTION

There is no unique method to solve nonlinear ordinary differential equations (ODE) as well as partial differential equations (PDE). Several techniques were developed and successfully applied by many scientists such as the Exp-function method [1,2], the tanh-function method [3,4], the homogeneous balance method [5,6], the (G'/G)-expansion method [7,8], the Backlund transformation method [9], the Jacobi elliptic function method [10], the modified simple equation method [11,12], the EMSEM[13].

In the sequel, we obtain exact solutions of Phi-4 PDE

$$u_{tt} - u_{xx} + m^2 u + \lambda u^3 = 0 \tag{1}$$

using the EMSEM.

Let us first describe the EMSEM in section II. In Section 3, we apply the method to Phi-4 PDE. Finally, we discuss the results in Section 4.

2. THE EMSEM

Suppose we have a nonlinear PDE of the form

$$F(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \dots) = 0 \tag{2}$$

where F is a polynomial of u and its partial derivatives. The method involves four steps.

Step 1: Using the transformation

$$u(x,t) = u(\zeta), \quad \zeta = p(t)x + q(t), \tag{3}$$

where $p(t)$ and $q(t)$ are differentiable functions of t, from (2) and (3) we get the following ODE

$$F(u, (p'x + q')u', pu', \dots) = 0 \tag{4}$$

Step 2: Suppose that equation (4) has the solution

$$u(\zeta) = \sum_{k=0}^{n} A_k(t)\left[\frac{\varphi'(\zeta)}{\varphi(\zeta)}\right]^k \tag{5}$$

where $A_k(t)$ are functions of t, $A_k(t)$ and $\varphi(\zeta)$ are unknown expressions to be obtained, $A_n \neq 0$.

Step 3: Compute n in equation (5). This is accomplished by balancing the highest order derivative and nonlinear term in equation (4).

Step 4: Substitute equation (5) into equation (4). Combine all the terms of the same power of $\varphi(\zeta)^{-j}$, where $j \geq 0$, and equate their coefficients to zero. This results in a system of algebraic and differential equations which can be solved to find $A_k(t)$ and $\varphi(\zeta)$. Consequently, we get a closed form solution of equation (2).

3. APPLICATION TO PHI-4 PDE

In this section, we solve the Phi-4 PDE

$$u_{tt} - u_{xx} + m^2 u + \lambda u^3 = 0$$

The transformation (3) reduces the above equation to:

$$u'(p''x + q'') + u''(p'x + q')^2 - u''p^2 + m^2 u + \lambda u^3 = 0 \tag{6}$$

Taking the s balance between u'' and u^3, we get $n = 1$. The solution of equation (6) has the form

$$u(\zeta) = A_0(t) + A_1(t)\left[\frac{\varphi'(\zeta)}{\varphi(\zeta)}\right] \tag{7}$$

Our goal is to solve for $A_0(t)$ and $A_1(t)$. We have

$$u' = A_1(t)\left(\frac{\varphi''}{\varphi} - \left(\frac{\varphi'}{\varphi}\right)^2\right) \tag{8}$$

$$u'' = A_1(t)\left(\frac{\varphi'''}{\varphi} - 3\frac{\varphi'\varphi''}{\varphi^2} + 2\left(\frac{\varphi'}{\varphi}\right)^3\right) \tag{9}$$

Substituting equations (7)-(9) in (6) and setting all the coefficients of φ^0, φ^{-1}, $\varphi^{-1}x$, $\varphi^{-1}x^2$, φ^{-2}, $\varphi^{-2}x$, $\varphi^{-2}x^2$ and φ^{-3} to zero, we get respectively

$$m^2 A_0 + \lambda A_0^3 = 0 \tag{10}$$

$$q''\varphi'' + (q'^2 - p^2)\varphi''' + (m^2 + 3\lambda A_0^2)\varphi' = 0$$

which, by integration with respect to ζ, reduces to

$$q''\varphi' + (q'^2 - p^2)\varphi'' + (m^2 + 3\lambda A_0^2)\varphi = 0 \tag{11}$$

$$p''\varphi'' + 2p'q'\varphi''' = 0$$

which, by integration with respect to ζ, reduces to

$$p''\varphi' + 2p'q'\varphi'' = 0 \tag{12}$$

$$p'^2\varphi''' = 0 \tag{13}$$

$$(3\lambda A_0 A_1 - q'')\varphi' - 3\left(q'^2 - p^2\right)\varphi'' = 0$$

which, by integration with respect to ζ, reduces to

$$(3\lambda A_0 A_1 - q'')\varphi - 3\left(q'^2 - p^2\right)\varphi' = 0 \tag{14}$$

$$p''\varphi' - 6p'q'\varphi'' = 0 \tag{15}$$

$$p'^2\varphi'\varphi'' = 0 \tag{16}$$

$$2\left(q'^2 - p^2\right) + \lambda A_1^2 = 0 \tag{17}$$

Various solutions could be obtained. From (10), $A_0 = 0$, $A_0 = \pm\frac{m}{\sqrt{\lambda}}$. From (13), or (16) it follows that $p(t) = k$.

From (17), $A_1 = \pm\sqrt{\frac{2(p^2-q'^2)}{\lambda}}$.

From (11) and (14), after eliminating φ , we get

$$\frac{\varphi''}{\varphi'} = \left(\frac{q''}{p^2 - q'^2} - \frac{3(m^2 + 3\lambda A_0^2)}{(-q'' + 3\lambda A_0 A_1)}\right) \tag{18}$$

Integrating equation (18) with respect to ζ yields

$$\varphi' = b_1(t)exp(r(t)\zeta) \tag{19}$$

where

$$r(t) = \frac{q''}{p^2 - q'^2} - \frac{3(m^2 + 3\lambda A_0^2)}{(-q'' + 3\lambda A_0 A_1)}$$

Therefore

$$\varphi = b_2(t) + \frac{b_1(t)}{r(t)}exp(r(t)\zeta) \tag{20}$$

From (7), (19), and (20), it follows that

$$u(x,t) = A_0 + A_1\left(\frac{b_1(t)exp(r(t)\zeta)}{b_2(t)+\frac{b_1(t)}{r(t)}exp(r(t)\zeta)}\right) \tag{21}$$

4. CONCLUSION

The EMSEM is applied successfully to Phi-4 PDE. Clearly, (21) is an additional set of solutions of the Phi-4 PDE to the ones derived in [11] using the modified simple equation method.

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