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Solution of Elastic Half Space Problem using Boussinesq Displacement Potential Functions

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ABSTRACT---- ***In this study, the Boussinesq displacement potential functions were used to solve the elastic half space problem involving a point load acting at the origin. Displacement field components were obtained from the Boussinesq potential functions using Love's expressions. Strain displacement and stress-strain laws were used simultaneously to obtain the stress fields from the displacement fields. It was found that solutions for stress and displacement fields were exactly the same expressions obtained by other researchers in the technical literature.***

Keywords--- Boussinesq displacement potential functions, elastic half space problem, Boussinesq problem, stress fields, displacement fields

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1. INTRODUCTION

The elastic half space problem is the problem of finding stress fields and displacement fields in a loaded medium bounded by a horizontal boundary surface, and which extends infinitely in both the radial and depth coordinate directions, where the depth coordinate is defined as pointing downwards, perpendicularly to the boundary surface. Such problems are frequently encountered in geotechnical engineering where the elastic medium is the soil mass, idealized as semi-infinite in extent [1, 2]. They are also found in the mathematical theory of elasticity. Specific examples of the elastic half space problem include the Boussinesq, Kelvin, Mellin and Cerrutti problems [2, 3, 4].

The formulation and solution of elastic half space problems belong to the mathematical theory of elasticity in three dimensions. The fundamental equations are the differential equations of equilibrium, the material constitutive laws, the kinematic relations, and those are solved subject to the boundary conditions [5, 6]. The elastic half space medium can be linear elastic or non linear elastic, isotropic or anisotropic, homogeneous or heterogeneous. In this study, the elastic half space medium is assumed to be linear elastic, homogeneous and isotropic, yielding simplification in the governing equations.

Two basic methods are commonly used in formulating elasticity problems; namely: displacement method and stress method. In displacement formulation, the governing equations are expressed such that unknown displacements are the primary variables, and stresses are eliminated from the equations [7, 8, 9]. In stress based formulation, displacements are eliminated from the governing equations, which we reformulated with stresses as the unknown primary variables. Stress based methods of elasticity problems have been presented by Beltrami-Michell for 3 dimensional problems as a system of six partial differential equations expressed in terms of the six components of stresses [10, 11]. Displacement based methods of elasticity problems have been presented by Navier, Lamé, as a system of three differential equations in terms of the three components of the displacement field [12, 13]. The simplification of the presentation of three dimensional elasticity problems have led to the development of stress and displacement functions that identically solve the stress based and the displacement based equations of elasticity theory [14, 15]. Some stress functions include: Airy stress functions, Morera stress functions, Maxwell stress functions, Michell stress functions. Displacement functions in elasticity theory include: Cerrutti functions, Boussinesq [5] functions, Green and Zerna [16] functions, Trefftz [17] functions, Boussinesq-Papkovich functions.

2. RESEARCH AIM AND OBJECTIVES

The aim of this study is to use the Boussinesq displacement functions to solve the elastic half space problem. The objectives include:

(i) to present the Boussinesq displacement functions in the general case; and then derive the Boussinesq displacement function for the specific case of elasic half space under point load applied at the origin (Boussinesq point load problem).

(ii) to find the Cartesian components of the displacement field from the Boussinesq displacement potential functions using relations obtained by Love.

(iii) to simultaneously use the strain displacement and the stress-strain equations to obtain the Cartesian components of the stress field in terms of the Boussinesq displacement potential functions.

3. THEORETICAL FRAMEWORK

The basic equations of the theory of elasticity that govern the elastic half space problem are the differential equations of equilibrium, the stress-strain laws, the strain displacement relations, the boundary conditions and the compatibility relations [1, 2, 3, 4, 6, 8, 12].

The differential equations of equilibrium when body forces are absent are [1, 7, 8, 11, 12, 13]:

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}=0 \quad (1)$$

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}=0 \quad (2)$$

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}=0 \quad (3)$$

$$\tau_{xy}=\tau_{yx} \quad (4)$$

$$\tau_{yz}=\tau_{zy} \quad (5)$$

$$\tau_{xz}=\tau_{zx} \quad (6)$$

where $\sigma_{xx},\sigma_{yy},\sigma_{zz}$ are the normal stresses, $\tau_{xy},\tau_{yz},$ and τ_{zx} are shear stresses.

The stress-strain laws are given by [1, 6, 8, 11, 12, 13]:

$$\sigma_{xx}=\lambda\varepsilon_v+2G\varepsilon_{xx} \quad (7)$$

$$\sigma_{yy}=\lambda\varepsilon_v+2G\varepsilon_{yy} \quad (8)$$

$$\sigma_{zz}=\lambda\varepsilon_v+2G\varepsilon_{zz} \quad (9)$$

$$\tau_{xy}=G\gamma_{xy} \quad (10)$$

$$\tau_{yz}=G\gamma_{yz} \quad (11)$$

$$\tau_{zx}=G\gamma_{zx} \quad (12)$$

$$\lambda=\frac{\mu E}{(1+\mu)(1-2\mu)} \quad (13)$$

$$G=\frac{E}{2(1+\mu)} \quad (14)$$

$$\varepsilon_v=\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}$$

where $\varepsilon_{xx},\varepsilon_{yy},\varepsilon_{zz}$ are normal strains, $\gamma_{xy},\gamma_{yz},\gamma_{zx}$ are shear strains, λ is the Lamé's constant, G is the shear modulus, ε_v is the volumetric strain, μ is the Poisson's ratio of the medium and E is the Young's modulus of elasticity.

The strain-displacement relations are given by [1, 6, 8, 11, 12, 13]

$$\varepsilon_{xx}=\frac{\partial u_x}{\partial x} \quad (15)$$

$$\varepsilon_{yy}=\frac{\partial u_y}{\partial y} \quad (16)$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{17}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \tag{18}$$

$$\gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \tag{19}$$

$$\gamma_{zx} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \tag{20}$$

where u_x, u_y, and u_z are the x, y, and z Cartesian components of the elastic displacement field.

4. METHODOLOGY

Let $p_z(x_1, x_2)$ denote the distribution of load in the z direction acting on a soil mass idealized as an elastic half space as shown in Figure 1. x_1 and x_2 are dummy variables introduced to avoid confusion with the x and y coordinate variables.

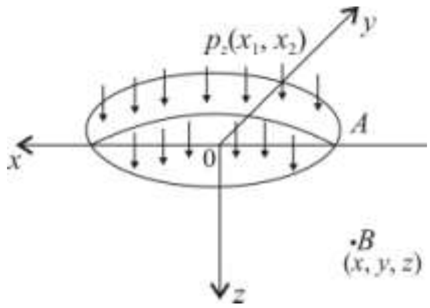

Figure 1

The load $p_z(x_1, x_2)$ is applied on an area, A, of the boundary surface, which is the xy coordinate plane (z = 0). For any arbitrary point $B(x, y, z)$ within the soil half space, and an arbitrary point $A_1(x_1, x_2)$ on the loaded region of the surface, the distance R is given by:

$$R^2 = (x_1 - x)^2 + (x_2 - y)^2 + (0 - z)^2 \tag{21}$$

$$R^2 = (x_1 - x)^2 + (x_2 - y)^2 + z^2 \tag{22}$$

Boussinesq's potential functions f_{B1}, f_{B2} and f_{B3} that satisfy the Laplace equation in three dimensional Cartesian coordinate space are given by:

$$f_{B_1} = \iint_A p(x_1, x_2)(z\ln(z+R) - R)dx_1dx_2 \tag{23}$$

$$f_{B_2} = \frac{\partial f_{B_1}}{\partial z} = \frac{\partial}{\partial z}\iint_A p(x_1, x_2)(z\ln(z+R) - R)dx_1dx_2 \tag{24}$$

$$f_{B_2} = \iint_A p(x_1, x_2)\ln(z+R)dx_1dx_2 \tag{25}$$

$$f_{B_3} = \frac{\partial f_{B_2}}{\partial z} = \frac{\partial}{\partial z}\iint_A p(x_1, x_2)\ln(z+R)dx_1dx_2 \tag{26}$$

$$f_{B_3} = \iint_A p(x_1, x_2)\frac{1}{R}dx_1dx_2 = \frac{\partial^2 f_{B_1}}{\partial z^2} \tag{27}$$

For point load, P, applied at the origin O, (x_1 = 0, x_2 = 0, z = 0) of the elastic half space, the load $p(x_1, x_2)$ can be represented using Dirac delta functions as:

$$p(x_1, x_2) = P\delta(x_1 = 0, x_2 = 0) \tag{28}$$

where $\delta(x_1 = 0, x_2 = 0)$ is the Dirac delta function.

The Boussinesq potential functions become for a point load *P* applied at the origin of the elastic half space:

$$f_{B_1} = \iint_A P\delta(x_1 = 0, x_2 = 0)(z\ln(z+R) - R)dx_1dx_2 \quad (29)$$

$$f_{B_1} = P(z\ln(z+R) - R) \quad (30)$$

$$f_{B_2} = \iint_A p\delta(x_1 = 0, x_2 = 0)\ln(z+R)dx_1dx_2 \quad (31)$$

$$f_{B_2} = P\ln(z+R) \quad (32)$$

$$f_{B_3} = \iint_A P\delta(x_1 = 0, x_2 = 0)\frac{1}{R}dx_1dx_2 \quad (33)$$

$$f_{B_3} = \frac{P}{R} \quad (34)$$

Love [18] showed that the elastic displacement fields are derivable from the Boussinesq potential functions as follows:

$$u_x = \frac{1}{4\pi G}\left\{-\frac{\partial f_{B_2}}{\partial x} + 2\mu\frac{\partial f_{B_2}}{\partial x} - z\frac{\partial f_{B_3}}{\partial x}\right\} \quad (35)$$

$$u_y = \frac{1}{4\pi G}\left\{-\frac{\partial f_{B_2}}{\partial y} + 2\mu\frac{\partial f_{B_2}}{\partial y} - z\frac{\partial f_{B_3}}{\partial y}\right\} \quad (36)$$

$$u_z = \frac{1}{4\pi G}\left\{\frac{\partial f_{B_2}}{\partial z} + (1-2\mu)f_{B_3} - z\frac{\partial f_{B_3}}{\partial z}\right\} \quad (37)$$

where u_x, u_y, and u_z are the *x*, *y*, and *z* Cartesian components of the elastic displacement field, *G* is the shear modulus of the elastic half space.
Rearranging,

$$u_x = -\frac{1}{4\pi G}\left((1-2\mu)\frac{\partial f_{B_2}}{\partial x} + z\frac{\partial f_{B_3}}{\partial x}\right) \quad (38)$$

$$u_y = -\frac{1}{4\pi G}\left((1-2\mu)\frac{\partial f_{B_2}}{\partial y} + z\frac{\partial f_{B_3}}{\partial y}\right) \quad (39)$$

$$u_z = \frac{1}{4\pi G}\left(2(1-\mu)f_{B_3} - z\frac{\partial f_{B_3}}{\partial x}\right) \quad (40)$$

The Boussinesq potential functions f_{B2} and f_{B3} both satisfy the Laplace equation in the space coordinates. Thus,

$$\nabla^2 f_{B_2} = 0 \quad (41)$$

$$\nabla^2 f_{B_3} = 0 \quad (42)$$

5. RESULTS

Using Love's expressions, the displacement components are found as:

$$u_x = -\frac{1}{4\pi G}\left\{(1-2\mu)\frac{\partial}{\partial x}P\ln(z+R) + z\frac{\partial}{\partial x}\frac{P}{R}\right\} \quad (43)$$

$$u_x = -\frac{P}{4\pi G}\left\{(1-2\mu)\frac{\partial}{\partial x}\ln(z+R) + z\frac{\partial}{\partial x}\left(\frac{1}{R}\right)\right\} \quad (44)$$

$$u_x = -\frac{P}{4\pi G}\left(\frac{(1-2\mu)x}{R(R+z)} - \frac{zx}{R^3}\right) \quad (45)$$

$$u_x = \frac{P}{4\pi G}\left(\frac{zx}{R^3} - \frac{(1-2\mu)x}{R(R+z)}\right) \quad (46)$$

$$u_y = -\frac{1}{4\pi G}\left((1-2\mu)\frac{\partial}{\partial y}P\ln(z+R) + z\frac{\partial}{\partial y}\left(\frac{P}{R}\right)\right) \tag{47}$$

$$u_y = -\frac{P}{4\pi G}\left((1-2\mu)\frac{\partial}{\partial y}\ln(z+R) + z\frac{\partial}{\partial y}\left(\frac{1}{R}\right)\right) \tag{48}$$

$$u_y = -\frac{P}{4\pi G}\left(\frac{(1-2\mu)y}{R(R+z)} - \frac{zy}{R^3}\right) \tag{49}$$

$$u_z = \frac{1}{4\pi G}\left(2(1-\mu)\frac{P}{R} - z\frac{\partial}{\partial z}\left(\frac{P}{R}\right)\right) \tag{50}$$

$$u_z = \frac{P}{4\pi G}\left(\frac{2(1-\mu)}{R} - z\frac{\partial}{\partial z}\left(\frac{1}{R}\right)\right) \tag{51}$$

$$u_z = \frac{P}{4\pi G}\left(\frac{z^2}{R^3} - \frac{2(1-\mu)}{R}\right) \tag{52}$$

The stress components are obtained, by simultaneous application of the strain-displacement laws as follows:

$$\sigma_{xx} = \frac{1}{2\pi}\left(2\mu\frac{\partial f_{B_3}}{\partial z} - z\frac{\partial^2 f_{B_3}}{\partial x^2} - (1-2\mu)\frac{\partial^2 f_{B_2}}{\partial x^2}\right) \tag{53}$$

$$\sigma_{yy} = \frac{1}{2\pi}\left(2\mu\frac{\partial f_{B_3}}{\partial z} - z\frac{\partial^2 f_{B_3}}{\partial y^2} - (1-2\mu)\frac{\partial^2 f_{B_2}}{\partial y^2}\right) \tag{54}$$

$$\sigma_{zz} = \frac{1}{2\pi}\left(\frac{\partial f_{B_3}}{\partial z} - z\frac{\partial^2 f_{B_3}}{\partial z^2}\right) \tag{55}$$

$$\tau_{xy} = -\frac{1}{2\pi}\left((1-2\mu)\frac{\partial^2 f_{B_2}}{\partial x\partial y} + z\frac{\partial^2 f_{B_3}}{\partial x\partial y}\right) \tag{56}$$

$$\tau_{yz} = -\frac{1}{2\pi}\left(z\frac{\partial^2 f_{B_3}}{\partial y\partial z}\right) \tag{57}$$

$$\tau_{zx} = -\frac{1}{2\pi}z\frac{\partial^2 f_{B_3}}{\partial x\partial z} \tag{58}$$

By evaluation of the partial differentiation, and substitution, we obtain upon simplification,

$$\sigma_{xx} = \frac{P}{2\pi}\left\{\frac{1-2\mu}{r^2}\left(\left(1-\frac{z}{R}\right)\frac{x^2-y^2}{r^2} + \frac{zy^2}{R^3}\right) - \frac{3x^2z}{R^5}\right\} \tag{59}$$

$$\sigma_{yy} = \frac{P}{2\pi}\left\{\frac{1-2\mu}{r^2}\left(\left(1-\frac{z}{R}\right)\frac{y^2-x^2}{r^2} + \frac{x^2z}{R^3}\right) - \frac{3y^2z}{R^5}\right\} \tag{60}$$

$$\sigma_{zz} = -\frac{3P}{2\pi}\frac{z^3}{R^5} \tag{61}$$

$$\tau_{xy} = \frac{P}{2\pi}\left\{\left(\frac{1-2\mu}{r^2}\right)\left(\left(1-\frac{z}{R}\right)\frac{xy}{r^2} - \frac{xyz}{R^3}\right) - \frac{3xyz}{R^5}\right\} \tag{62}$$

$$\tau_{xz} = -\frac{3P}{2\pi}\frac{xz^2}{R^5} \tag{63}$$

$$\tau_{yz} = -\frac{3P}{2\pi}\frac{yz^2}{R^5} \tag{64}$$

where $r^2 = x^2 + y^2$ (65)

Alternatively,

$$\sigma_{xx} = \frac{-P}{2\pi R^2}\left\{\frac{3x^2 z}{R^3} - (1-2\mu)\left(\frac{z}{R} - \frac{R}{R+z} + \frac{x^2(2R+z)}{R(R+z)^2}\right)\right\} \quad (66)$$

$$\sigma_{yy} = \frac{-P}{2\pi R^2}\left\{\frac{3y^2 z}{R^3} - (1-2\mu)\left(\frac{x}{R} - \frac{R}{R+z} + \frac{y^2(2R+z)}{R(R+z)^2}\right)\right\} \quad (67)$$

$$\tau_{xy} = \frac{-Pxy}{2\pi R^2}\left(\frac{3z}{R^3} - \frac{(1-2\mu)(2R+z)}{R(R+z)^2}\right) \quad (68)$$

Displacement on the surface (*z* = 0)

The Cartesian components of the displacement field on the surface ($z = 0$) are from Equations (46), (49) and (52), given by:

$$u_x = -\frac{P(1-2\mu)x}{4\pi G r^2} \quad (69)$$

$$u_y = -\frac{P(1-2\mu)y}{4\pi G r^2} \quad (70)$$

$$u_z = \frac{2(1-\mu)P}{4\pi G r} \quad (71)$$

At the point of application of the point load, $r = 0$, $R = 0$, and the expressions for displacement components are singular and hence indeterminate. The stresses are also indeterminate at the point of application of the point load as the expressions for stresses become singular at the origin.

6. DISCUSSION

In this study, the Boussinesq displacement potential functions have been successfully implemented to solve for stress fields and displacement fields in an elastic half space due to point load applied at the origin. The Boussinesq displacement potential functions were presented in general for distributed normal load on the elastic half space as Equations (23), (24) and (26). For point load applied at the origin, the Boussinesq displacement potential functions were found using Dirac delta function theory as Equations (30), (32) and (34). The displacement field components were obtained, using Love's expressions as Equations (38), (39) and (40) for the general elastic half space problem. For the specific case of Boussinesq elastic half space problem, the displacement field components were obtained as Equations (46), (49) and (52). Strain displacement relations for small displacement, isotropic three dimensional elasticity and stress-strain laws were simultaneously used to find the stresses in terms of the Boussinesq displacement potential function as Equations (53) – (58). For the Boussinesq problem, the stress fields were obtained as Equations (59) – (64). It was observed that the displacement fields and stress fields obtained were exactly the same expressions as displacement and stress fields obtained by Nwoji et al [1] using Green and Zerna displacement potential functions; Ike et al [4] using Trefftz potential functions, and, Ike et al [3] using Bessel functions in a stress based formulation.

7. CONCLUSIONS

The following conclusions can be drawn from this study:

(i) The Boussinesq displacement potential functions have been successfully implemented in deriving the stress fields and displacement field components in a soil mass that is linear elastic, homogeneous, isotropic and of semi-infinite extent $(-\infty \le x \le \infty, -\infty \le y \le \infty, 0 \le z \le \infty)$

(ii) The Boussinesq displacement potential functions simplify the problem of three dimensional elasticity involving the elastic homogeneous half space to the problem of partial differentiation of the Boussinesq potential functions in order to obtain displacement components and stress fields.

(iii) The solutions obtained for the stress fields and displacement field components are valid and true at all points in the elastic half space, except at the point of application of the point load where the expressions for the stresses and displacements become singular and indeterminate.

8. REFERENCES

[1] Nwoji C.U., Onah H.N., Mama B.O. and Ike C.C. Solution of the Boussinesq problem of half space using Green and Zerna displacement potential function method. Electronic Journal of Geotechnical Engineering (EJGE), 2017, Volume 22, Bundle 11 (22.11) pp. 4305-4314. Available at ejge.com

[2] Ike C.C. First principles derivation of a stress function for axially symmetric elasticity problems, and application to Boussinesq problem. Nigerian Journal of Technology (NIJOTECH), Vol. 36, No.3, July 2017, pp. 767-772.

[3] Ike C.C., Onah H.N. and Nwoji C.U. Bessel functions for axisymmetric elasticity problems of the elastic half space soil, a potential function method. Nigerian Journal of Technology (NIJOTECH), Vol. 36, No.3, July 2017, pp. 773-781.

[4] Ike C.C., Mama B.O., Onah H.N. and Nwoji C.U. Trefftz harmonic function method for solving Boussinesq problem. Electronic Journal of Geotechnical Engineering (EJGE) 2017 (22.12), pp. 4589-4601. Available at ejge.com

[5] Boussinesq J. Application des potentials a l'etude de l'equilibre et des mouvements des solides elastiques. Paris, Ganther-Villars, 1885.

[6] Barber J.R. Elasticity, 3rd Revised Edition. Springer Science and Business Media Dordrecht the Netherlands, 2010.

[7] Kachanov M.L., Shafiro B. and Tsukrov I. Handbook of Elasticity Solutions. Springer Science and Business Media. Kluwer Academic Publishers Dordrencht The Netherlands, 2003.

[8] Sadd M.H. Elasticity Theory, Application and Numerics. University of Rhode Island Elsevier https://personalegr.uri.edu/teggart/course/ce571/chapter7.pptx.

[9] Sithram T.G. and Govinda Reju L. Applied Elasicity for Engineers Module Elastic Solutions and Applications in Geomechanics 14.139.172.204/nptel/1/CSE/web/105108070/modulue 8/lecture 17.pdf

[10] Palaniappan D. A general solution of equations of equilibrium in linear elasticity. Applied Mathematical modelling, 35 (2011) pp. 5494-5499, Elsevier.

[11] Westergaard H.M. Theory of Elasticity and Plasticity. Wiley New York, 1952.

[12] Sokolnikoff I.S. Mathematical Theory of Elasticity, Second Edition. Data McGraw Hill Publishing Company Ltd, Bombay New Delhi, 1956.

[13] Timoshenko S.P. and Goodier J.N. Theory of Elasticity, Third Edition, McGraw Hill New York, 1970

[14] Hazel A. MATH 350211 Elasticity. www.maths.manchester.acuk/_a hazel/MATHS. Nov 30, 2015.

[15] Abeyartne R. Continuum Mechanics Volume II of Lecture notes on the mechanics of elastic solids, Cambridge http//web.mit.edu/abeyartne/lecture_notes.html. 11, May, 2012.

[16] Green A.E. and Zerna W. Theoretical Elasticity. Oxford University Press, London, 1954.

[17] Trefftz E. Mathematische Elastizitate theorie in Handbuch der Physik, Ed VI, Berlin, Springer, 1928.

[18] Love A.E.H. Mathematical theory of elasticity, 4th Edition, Dover Publications Inc. New York, 1944.