P-coretractable and Strongly P-coretractable Modules

Inaam Mohammed Ali Hadi[1] and Shukur Neamah Al-aeashi [2,*]

[1] Department Of Mathematics, College 0f Education for Pure Sciences (Ibn-Al-Haitham)
University Of Baghdad , Iraq

[2] Department Of Urban Planning , College 0f Physical Planning
University Of Kufa , Iraq

[*]*Corresponding author's email: shukur.mobred [AT] uokufa.edu.iq*

ABSTRACT— ***In this paper, we introduce the notion of P-coretractable module . Some basic properties of this class of modules are investigated and some relationships between these modules and other related concepts are introduced . Also , we give the notion of strongly P-coretractable and study it comparison with P-coretractable , moreover the mono-P-coretractable concept are introduced and studied .***

Keywords— (strongly) coretractable modules, (strongly) P-coretractable , mono-coretractable , mono-P-coretractable , purely-Rickart .

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1. INTRODUCTION

Throughout this paper all rings have identities and all modules are unital right R-modules . A module M is called coretractable if for each a proper submodule N of M, there exists a nonzero R-homomorphism f:M/N→M [1] , and M is called strongly coretractable module if for each proper submodule N of M , there exists a nonzero R-homomorphism f:M/N→M such that Imf+N=M [2],[8] . It is clear every strongly coretractable module is coretractable but it is not conversely. This work consists of three sections , in section one , we introduce the notion of purely-coretractable or P-coretractable module where an R-module M is called P-coretractable if for each proper pure submodule N of M , there exists a nonzero homomorphism $f\in Hom_R(M/N,M)$ also we give some examples and remarks about it . Some basic properties of P-coretractable modules are given . In section two , we introduce the notion of mono-P-coretractable . In section three , we introduce and study the notion of strongly P-coretractable module and we compare its properties with properties of P-coretractable module .

2. PURELY-CORETRACTABLE (BRIEFLY P-CORETRACTABLE)

In this section, we introduce the concept of P-coretractable module and give some properties of this class module . In the beginning , we recall a submodule N of an R-module M is a pure , if for every finitely generated ideal I of R , $IM\cap N=IN$ [3] .

Definition(1.1): An R-module is called purely-coretractable (Briefly P-coretractable) if for each proper pure submodule N of M, there exists a nonzero homomorphism $f\in Hom(M/N, M)$.

Equivalently , M is a P-coretractable module if for each proper pure submodule N of M , there exists $g\in End_R(M)$, $g\neq 0$ and g(N)=0 . A ring R is called P-coretractable if R is P-coretractable R-module .

Examples and Remark(1.2):

(1) Every coretractable module is P-coretractable . But the converse is not true in general and we shall give an example later after Corollary(1.13) .

(2) Every semisimple module is P-coretractable module but the converse is not true in general . For example , $M=Z_4$ as Z-module is not semisimple module , but M is P-coretractable module since 0 is the only proper pure submodule of M.

(3) Every pure simple R-module is a P-coretractable . Where an R-module M is called pure simple module if the only two pure submodules are 0 and M [4] .

The converse of this Remark is not true in general . For example Consider $M=Z_2\oplus Z_2$ as Z-module is P-coretractable module but not pure simple module .

(4) Every pure split module is P-coretractable module . Where an R-module M is called pure split if every pure submodule is a direct summand of M [5] . In particular Z_8Z_2 is pure split , so it is P-coretractable .

(5) Let P be the set of all prime numbers and M=$\prod_{p\in P} Zp$ (That is M=$Z_2\times Z_3\times Z_5\times\ldots$) . Then we shall show that M is not P-coretractable module . Let K= $\bigoplus_{p\in P} Z_p$, then K is a pure submodule of M by [6, Lam. 4.84(d)] . Then by using the same argument of proof of [1,Example(2.9)] . Let f∈Hom$_R$(M/K,Z_p) for some p∈P , M/K = PM+K/K=P(M/K)⊆kerf , so f=0 . Hence Hom(M/K,M) ≅ $\prod_{p\in P} \mathrm{Hom}\left(\frac{M}{K}, Z_p\right)$ = 0 , then M is not P-coretractable module . However Z_p (for any p ∈ P) is P-coretractable module .
(6) Let M ≅M' where M is a P-coretractable R-module . Then M' is a P-coretractable R-module .

Proof : we shall introduce the proof in section three by more generally. See Proposition(3.3) .
(7) P-coretractability is not preserved by taking submodules , factor modules and direct summands since for any R-module M and a cogenerator R-module C , C⊕ M is a cogenerator and so is a coretractable module and so P-coretractable , but M need not be coretractable module .
(8) Let R be a PID , M is an R-module . If M is C-coretractable , then M is P-coretractable . Where an R-module M is called C-coretractable module if for each proper closed submodule N of M , there exists a nonzero homomorphism f∈Hom$_R$(M/N,M) . A ring R is called C-cortractable if R is C- cortractable R-module [8] .

Proof : Let N be a proper pure submodule of M . By [6,Lam, Exc.15,P.242], N is closed . As M is C-coretractable , then there exists f ∈End$_R$(M) , f≠0 and f(N)=0. Thus M is P-coretractable .
(9) Let R be a PID , M is an R-module . If M is coquasi-Dedkend , then M is P-coretractable . Where an R-module M is coquasi-Dedekind module if for each f∈End$_R$(M), f≠0 , f is an epimomorphism . [7 ,Theorem(2.1.4) ,P.33] .

Proof: By [7,Theorem(2.1.15)] , M has no proper nonzero pure submodule , that is M is pure simple . Thus M is P-coretractable by part (3)
(10) Let M=Z_{p^∞} as Z-module is P-coretractable and injective . Then M=$\bigoplus_{i\in I}$M$_i$ (M$_i$=M) is P-coretractable since M is C-coretractable by Example (2) after Theorem(1.10)) in [8] , and Z is a PID . But M is not coquasi-Dedekind .

Recall that an R-module M is called purely extending if every submodule is essential in pure submodule . Equivalently , M is purely extending if and only if every closed submodule is pure in M [9] .
Proposition*(1.3):* Let M be a purely extending R-module , if M is a P-coretractable, then M is a C-coretractable module .
Proof: Let K be a proper closed submodule of M . Since M is a purely extending module , then K is pure submodule . But M is a P-coretractable module , so there exists f∈End$_R$(M), f≠0 f(K)=0, then M is a C-coretractable module .
Recall that an R-module M is said to be regular (or F-regular) if R/ann(x) is regular ring for all nonzero x∈M [10,P.29] .
Equivalently , an R-module M is said to be regular (F-regular) if every submodule of M is a pure submodule [10,Theorem (1.7), P.35] .
Corollary*(1.4):* Let M be an F-regular R-module , then if M is a P-coretractable module , then M is a C-coretractable module .
Proof: It is clear since every F-regular is purely extending module and hence by Proposition(1.3) the result holds .
Proposition*(1.5):* Let M be an F-regular R-module , then M is a coretractable if and only if M is a P-coretractable .
Proof : (⟹) It is clear .
(⟸) Let N be a proper submodule of M . Since M is F-regular module , so N is pure submodule . But M is P-coretractable module , hence there exists f∈ End$_R$(M) , f ≠ 0 and f(N)= 0 , then M is coretractable module .
Recall that an R-module M is called a purely lifting if for every submodule N of M , there exists a pure submodule K of M such that K⊆N and N/K is small in M/K [11]. An R-module M is called a V-module if for every factor module N of M , Rad(N)=0 [12] .
Corollary*(1.6)*: If M is a purely lifting V-module . Then M is a P-coretractable if and only if M is a coretractable module .
Proof *:* Since M is V-module and M is purely lifting . Then M is an F-regular module [11,Proposition(2.2.4),P.40]. Hence we get the results by Proposition(1.5) .
Recall that an R-module M is called quasi-Dedekind if every proper nonzero submodule N of M is quasi-invertible where a submodule N of M is called quasi-invertible if Hom$_R$(M/N,M)=0 [13] . A nonzero ideal (right ideal) I of a ring R is quasi-invertable ideal (right ideal) of R if I is quasi-invertable submodule of R . Also M is a quasi-Dedekind R-module if for any nonzero f ∈ End$_R$(M) , f is monomorphism ; that is kerf= (0) [13,Theorem(1.5) , P.26] .
Proposition*(1.7)*: Let M be a P-coretractable quasi-Dedekind R-module. Then M is a pure simple .
Proof : Let N be a proper pure submodule of M .Then there exists f∈End$_R$(M), f≠0 and f(N)=0. As M is quasi-Dedekind module , hence f is monomorphism . Thus N=0 and hence M is pure simple module.
Recall that an R-module M is called purely Rickart if for all f∈End$_R$(M) , kerf is pure submodule of M [14] .

Theorem*(1.8):* Let M be a purely Rickart R-module . Then M is coretractable module if and only if for all proper submodule K of M , there exists a pure submodule W of M such that $K\subseteq W$ and M is P-coretractable.

Proof : ($\Longrightarrow$) Clear that M is P-coretractable module because M is coretractable module . Now, let K be a proper submodule of M . Since M is coretractable module , then there exists a nonzero R-homomorphism $f:M\rightarrow M$, $f(K)=0$, so $K\subseteq kerf$. But M is purely Rickart module , so kerf pure submodule of M . As $f\neq 0$ hence $kerf\neq M$ and hence kerf is a proper pure submodule such that $K\subseteq W\subset M$ (Where W=kerf) .

($\Longleftarrow$) Let K be a proper submodule of M . By hypothesis there exists a pure submodule W of M such that $K\subseteq W$. Since M is P-coretractable , hence $f\in End_R(M)$ such that $f(W)=0$,$f\neq 0$ implies to $f(K)=0$. Then M is coretractable module .

Recall that , let M be a right R-module and $S = End_R(M)$. Then M is said to be dual-purely Rickart (shortly, dual purely Rickart) module if the image in M of any single element of S is pure in M . That is for each $\alpha \in S$, $Im\alpha$ is pure submodule in M [14] .

Proposition*(1.9):* Let M be a mono-coretractable R-module . Then M is dual purely Rickart module , if M is purely Rickart module.

Proof : Let $f\in End_R(M)$. Since M is mono-coretractable module , then there exists $g\in End_R(M)$ such that Imf=kerg . As M is purely Rickart module , if kerg is pure submodule of M . Thus Imf is pure submodule of M and so M is dual purely Rickart module .

Recall that an R-module M is called finitely presented if any finite generated submodule of M is direct summand [6] .

Proposition*(1.10):* Let M be a Noetherian finitely presented R-module. Then M is a P-coretractable module .

Proof : Let N be a proper pure submodule of M . Since M is Noetherian module . Then N is finitely generated . As M is finitely presented , so N is direct summand submodule by [6,Lam.Exc.32,P.163]. Then $N \oplus W=M$ for some a submodule W of M , so $M/N\cong W$. Then M is P-coretractable .

Proposition*(1.11):*Let M be a Noetherian projective R-module . Then M is a P-coretractable module.

Proof : Let N be a proper pure submodule of M . Since M is Noetherian module , N is finitely generated . Hence by [15], N is a direct summand , then $M=N\oplus W$ for some a submodule W of M , then there exists an isomorphism $f:M/N\rightarrow W$ and let $i:W\rightarrow M$ be the inclusion map , therefore $i\circ f:M/N\rightarrow M$, $i\circ f\neq 0$. Thus M is P-coretractable module .

Now , we can present an example of P-coretractable but not coretractable .

Example*(1.12):* Consider $M=Z\oplus Z$ as Z-module , M is Noetherian and projective and so M is P-coretractable by Proposition(1.11) , but M is not coretractable module .

The following result follows by Proposition(1.11) , Since R is projective

Corollary*(1.13):* Let R be a Noetherian ring . Then R is a P-coretractable ring . The ring of integers Z is Noetherian , so It is a P-coretractable , but Z is not coretractable ring . Recall that a submodule N of an R-module M is called fully invariant if f(N) is contained in N for every R-endomorphism f of M [16] and a submodule N of an R-module M is called stable if for each $f \in Hom(N,M)$, $f(N)\subseteq N$ where an R-module M is called fully stable if every submodule of M is stable [17] .

Proposition*(1.14):*Let N be a direct summand submodule of a P-coretractable R-module M . If N is fully invariant submodule of M , then N is P-coretractable .

Proof : Since N is a direct summand submodule , so there exists a submodule W of M such that $N\oplus W=M$. Let K be a proper pure submodule of N , we have $K\oplus W$ is apure submodule in $N\oplus W=M$ (Since K is pure in N and W is pure in W) . Since M is a P-coretractable module, so there exists $f\in End_R(M)$, $f\neq 0$, $f(K\oplus W) =0$. suppose that g is the restriction map of f from N into M , $g\neq 0$. Also N is fully invariant direct summand . Then N is stable submodule . So $g(N)\subseteq N$. Therefore $g\in End_R(N)$, $g\neq 0$. $g(K)=f_N(K)=0$ .Thus N is P-coretractable module .

Corollary*(1.15):* Let N be a direct summand submodule of a P-coretractable and duo R-module M , then N is P-coretractable . where a module M is duo if every submodule is fully invariant [12] .

Proof : It is clear since every submodule if fully invariant in duo module .

Recall that an R-module M is called cogenerator if for every nonzero homomorphism $f:M_1\rightarrow M_2$ where M_1 and M_2 are R-modules , there exists $g:M_2\rightarrow M$ such that $g\circ f \neq 0$ [6, P.507] and [3, P.53] .

Proposition*(1.16):* Let N be a direct summand of a P-coretractable module M . If N is cogenerates M . Then N is P-coretractable module .

Proof : Suppose N is cogenerates M , so there exists $g\in Hom_R(M,N)$, $g\neq 0$. Let K be a pure submodule of N . Since N is direct summand of M, then $N\oplus W=M$ for some a submodule W of M . So $K\oplus W$ is pure in $N\oplus W=M$. Then there exists $f \in End_R(M)$, $f \neq 0$, $f(K\oplus W)=0$. Hence $g\circ f\neq 0$, Let h be a restriction map of $g\circ f$ on N , so $h \in End_R(N)$ and $h(K)=g(f(K))=0$. Therefore N is P-coretractable module .

For an R-module M . Recall that a module M has the pure intersection property (briefly PIP) if the intersection of any two pure submodules is again pure [9] .

Theorem*(1.17)* : Let { M_α : $\alpha\in I$ } be a family of P-coretractable R-module if for any α , $\beta \in I$, M_α is M_β–injective and $M=\bigoplus_{\alpha\in I} M_\alpha$ has PIP , then M is P-coretractable . In particular , if M is quasi-injective P- coretractable and satisfy PIP , then = $\bigoplus_{\alpha\in I} M_\alpha$ is P-coretractable for any index I , M_α =M for all $\alpha\in I$.

Proof : Let K be a proper pure submodule of M , then there exists $\beta \in I$ such that $M_\beta \not\subseteq K$. Since K is pure in M and M_β is pure in M and M satisfies PIP, so $K \cap M\beta \subset M_\beta$, and it is a proper pure submodule in M_β.Therefore there exists a nonzero homomorphism f: $M_\beta / K \cap M_\beta \to M_\beta$ and let $g: M_\beta/(K \cap M_\beta) \to M/K$ (Which is defined by $g(x+(K \cap M\beta))=x+K$ for all $x \in M\beta \subset M$) , then g is a monomorphism . As M_β is M_α-injective for any $\alpha \in I$ by hypothesis , M_β is M/K–injective by [3, proposition16.13], so there exists h: $M/K \to M_\beta$ such that $h \circ g = f$. Hence $0 \neq i \circ h \in Hom_R(M/K,M)$, where i: $M_\beta \to M$ is the natural inclusion .

Theorem*(1.18):* Let $M= \bigoplus_{\alpha \in I} M_\alpha$ such that M_α be a P-coretractable module $\alpha \in I$. If every pure submodule in M is fully invariant , then M is P-coretractable module .

Proof : Let N be a proper pure submodule of M . By hypothesis N is fully invariant , $N=\bigoplus_{\alpha \in I}(N \cap M_\alpha)$. Put $N \cap M_\alpha = N_\alpha$ for all $\alpha \in I$, since $N \cap N_\alpha \leq^{\oplus} N$, then N_α is pure submodule in N , but N is pure in M , then N_α pure in M . As $N_\alpha \leq M_\alpha$, we get N_α is pure in M_α . Also since N is a proper submodule of M , there exists at least one $\alpha_i \in I$, N_{α_i} proper submodule of M_{α_i} . But M_{α_i} is P-coretractable , so there exists f_{α_i}: $M_\alpha / N_\alpha \to M_\alpha$ and $f_\alpha \neq 0$. As $M/N \cong \bigoplus_{\alpha \in I}(M_\alpha / N_\alpha)$. Define h:$M/N \to M_{\alpha_i}$ by $h(m+N)= f_{\alpha_i}(m_{\alpha_i}+N_{\alpha_i})$ for any $m=\bigoplus_{\alpha \in I} m_\alpha \in M$. Then $h \neq 0$ and $g=i \circ h: M/N \to M$, $g \neq 0$.

3. MONO-P-CORETRACTABLE MODULES

In this section , we introduce the notion of mono-P-coretractable module and study some properties of this class module .

Definition*(2.1)*: An an R-module M is called mono-P-coretractable if for all a proper pure submodule of M , there exists $f \in End_R(M)$, $f \neq 0$ and N=kerf . Equivalently , A module M is mono-P-coretractable if for each proper pure submodule N of M , there exists a monomorphism f from M/N into M .

Recall that an R-module M is called co-epi-retractable if it contains a copy of any of its factor modules [18] . However, for more convenient , we call it mono-coretractable module.

Examples and Remarks (2.2):

(1) Every pure split module is a mono-P- coretractable .
(2) Every mono-coretractable module is mono-P-coretractable .
(3) Every pure simple module is mono-P-coretractable .
(4) Every mono-P-coretractable module is P-coretractable .
(5) Every semisimple module is mono-coretractable and hence it is mono-P-coretractable module by part(3) .
(6) Let M be an R-module . If M is a quasi-Dedekind mono-P-coretractable , then M is a pure simple .

Proof : Let N be a proper pure submodule of M . Since M is mono-P- coretractable, so there exists $f \in End_R(M)$, $f \neq 0$, f(N)=0 and N=kerf , but M is quasi-Dedekind , hence kerf=0 . Thus N=0 and hence M is pure simple module .

Let M be a right R-module and let $S = End_R(M)$. Recall that an R-module M is called a Rickart module if the right annihilator in M of any single element of S is generated by an idempotent of S . Equivalently , M is called Rickart module if for all $f \in S$, kerf $\leq^{\oplus}$M [19].

Proposition*(2.3):* Let M be a Rickart R-module . Then M is a mono-P-coretractable if and only if M is a pure split .

Proof : ($\Longrightarrow$) Let N be a proper pure submodule of M . Since M is mono-P- coretractable , then there exists $f \in End_R(M)$, $f \neq 0$ and N=kerf , but M is a Rickart , hence kerf is a direct summand of M for each $f \in End_R(M)$ and so N is direct summand of M . Thus M is pure split module .

($\Longleftarrow$) It follows by Examples and Remarks(2.2 (1)) .

Recall that an R-module M is called a strongly Rickart if and only if kerf = r_M(f) is a fully invariant direct summand in M for all $f \in End_R(M)$ [20] .

We introduce the following definition : An R-module M is called P-fully stable if every pure submodule is stable . It is clear that every fully stable is P-fully stable but not conversely .

Proposition*(2.4):* Let M be a strongly Rickart R-module . Then M is a mono-P-coretractable if and only if a P-fully stable and pure split .

Proof : ($\Longrightarrow$) Let N be a proper pure submodule of M . Since M is mono-P- coretractable , then there exists $f \in End_R(M)$, $f \neq 0$ and N=kerf , but M is a strongly Rickart , hence kerf is a stable direct summand of M for each $f \in End_R(M)$ and hence N is a stable direct summand of M . Thus M is P-fully stable pure split module .

($\Longleftarrow$) It is clear .

Recall that an R-module M is called mono-C-coretractable if for each proper closed submodule of M , there exists $f \in End_R(M)$, $f \neq 0$ and N=kerf [8] .

Proposition*(2.5)*:Let M be a purely extending . If M is a mono-P-coretractable module , then M is a mono-C-coretractable .

Proof : It is clear since if N is a proper closed submodule of M , then N is a pure . As M is a mono-P-coretractable , so there exists $f \in End_R(M)$, $f \neq 0$, f(K)=0 and K=kerf and hence M is a mono-C-coretractable .

Proposition(2.6):Let M be a mono-P-coretractable and P-fully stable module . Then every nonzero pure submodule of M is also mono-P-coretractable .

Proof : Suppose that N is a nonzero pure submodule of M . Let K be a proper pure submodule of N , so K is pure submodule of M . But M is mono-P-coretractable module . Then there exists $f \in End_R(M)$, $f \neq 0$, f(K)=0 and K=kerf , so if f(N)=0 , then $N \subseteq$ kerf =K so N=K which is a contradiction . Thus f(N) $\neq 0$. Let g be a restriction map from N into M . Since M is P-fully stable , so g(N) $\subseteq$N . Hence $g \in End_R(N)$ and $g \neq 0$ since $g(N)=f(N) \neq 0$. Hence g(K)=f(K)=0 . Thus $K \subseteq$ kerg $\subseteq$kerf =K . Then K= kerg .

4. STRONGLY-P-CORETRACTABLE MODULES

In this section, we define a new concept concerned directly with pure submodule called strongly P-coretractable module as one of generalization the concept strongly coretractable module see [17], also we introduce some properties and related with this concept .

Definition(3.1): An R-module is called strongly P-coretractable module if for each proper pure submodule K of M , there exists a nonzero homomorphism $f \in$Hom(M/K,M) and f(M/K)+K=M . Equivalently , M is strongly P-coretractable R-module if for each proper pure submodule K of M , there exists $g \in End_R(M)$, g(M/K)+K=M, $g \neq 0$ and g(K)=0 . A ring R is called strongly P-coretractable if R is strongly P-coretractable R-module .

Examples and Remarks(3.2):

(1) Every strongly coretractable is a strongly P-coretractable module but the converse is not true in general , for example the Z-module Z_4 is strongly P-coretractable , but it is not strongly coretractable , where an R-module M is called strongly coretractable module if for each proper submodule N of M , there exists a nonzero R-homomorphism $f:M/N \rightarrow M$ such that Imf+N=M [2],[8] .

(2) Every semisimple module is a strongly coretractable and hence strongly P-coretractable .

(3) Every pure simple R-module is a strongly P-coretractable module .

(4) Every pure split module is a strongly P-coretractable module .

(5) Every strongly P-coretractable module is a P-coretractable .

Proposition(3.3): Let $M \cong M'$, where M is a strongly P-coretractable R-module . Then M' is a strongly P-coretractable R-module.

Proof *:* Since $M \cong M'$, so there exists $f:M \longrightarrow M'$ be R-isomorphism . Let W be a proper pure submodule of M' . Then $N=f^{-1}(W)$ is proper pure submodule of M . Since M is strongly P-coretractable module, so there exists a nonzero R-homomorphism $h:M/N \longrightarrow M$ such that h(M/N)+N=M .

Define $g:M'/W \longrightarrow M'$, $g(f(m)+f(N))= f(m_1)$ where $h(m+N)=m_1 \in M$. To prove g is well-defined , suppose that f(m)+f(N)=f(x)+f(N) where $m,x \in M$. Then $f(m)-f(x) \in f(N)$, so $f(m-x) \in f(N)$ and so $m-x \in N$. Then m +N = x+N . Therefore $h(m+N)= m_1= m_2= h(x+N)$ (Since h is well-defined) which implies $g(f(m)+f(N))= f(m_1)=f(m_2) = g(f(x)+f(N))$ (Since f is well-defined) . Therefore g is well-defined , also g is an R-homomorphism .

To prove g(M'/W) +W = M' = f(M) . Let $m' \in M'$, then m'=f(m) for some $m \in M$. But $m=h(m_1+N)+n_1$ for some $m_1 \in M$ and $n_1 \in N$. Let $h(m_1+N)=m_2$, so $m=m_2+n_1$. But $g(f(m)+f(N))+f(n_1)= f(m_2)+f(n_1)= f(m_2+n_1) = f(m) =m'$. Therefore M' =Img+W, we get M' is a strongly P-coretractable R-module .

Proposition(3.4): Let M be a strongly P-coretractable R-module and N be a proper pure submodule of M ,then M/N is a strongly P-coretractable module .

Proof *:* Let W/N be a proper pure submodule of M/N . Since N is pure submodule of M , so W is pure submodule of M . But M is strongly P-coretractable module Then there exists a nonzero R-homomorphism $g:M/W \longrightarrow M$ such that Img+W=M . But $(M/N)/(W/N) \cong M/W$. Set $f= \pi \circ g$ where π is the natural epimorphism from M into M/W . Then $f(\frac{M}{W}) + \frac{W}{N} = \pi(g(\frac{M}{W})) + \frac{W}{N} = \frac{g(\frac{M}{W})+N}{N} + \frac{W}{N} = \frac{g(\frac{M}{W})+N+W}{N} = \frac{M+N}{N} = \frac{M}{N}$, and f $\neq 0$ (because if f is a zero mapping , then M/N=W/N which is a contradiction) , we can get M/N is also strongly P-coretractable .

Corollary(3.5): Let M be an R-module . If M is a strongly P-coretractable module. Then any direct summand submodule of M is a strongly P-coretractable module .

Proof *:* Since N is direct summand submodule of M , so there exists W is pure submodule of M such that $N \oplus W=M$. Thus M/W is strongly P-coretractable module by Proposition(3.4) . But $M/W \cong N$ so that N is also strongly P-coretractable module by Proposition(3.3)

Proposition(3.6): Let $M=M_1 \oplus M_2$, where M is duo module (or distributive or $annM_1+annM_2 = R$) . Then M is a strongly P-coretractable module if and only if M_1 and M_2 are strongly P-coretractable modules .

Proof *:* ($\Longrightarrow$) It follows directly by Corollary(3.5) .

($\Longleftarrow$) Let N be a proper pure submodule of M . Since M is duo (or distributive or $annM_1+annM_2 = R$) , then $N=(N \cap M_1) \oplus (N \cap M_2)$. Thus $N=N_1 \oplus N_2$ for some $N_1 \leq M_1$ and $N_2 \leq M_2$. Thus each of N_1 and N_2 are pure submodules in M_2 and M_2

respectively . Thus By the same argument proof of Theorem(2.7) in [2] , we can get M is a strongly P-coretractable module .

Compare the following Propositions with Proposition(1.3) , Proposition(1.5) and Proposition(1.8) respectively .

Proposition*(3.7):* Let M be a purely extending R-module , if M is strongly P-coretractable module , then M is strongly C-coretractable .

Proposition*(3.8):* Let M be an F-regular R-module , then M is strongly coretractable module if and only if M is strongly P-coretractable module .

Proposition*(3.9):* Let M be a purely Rickart R-module . Then M is strongly coretractable module if and only if for all proper submodule K of M , there exists a pure submodule W of M such that $K \subseteq W$ and M is strongly P-coretractable .

Proposition*(3.10):* Let M be a Noetherian finitely presented R-module. Then M is a strongly P-coretractable module .

Proof : Let N be a proper pure submodule of M . Since M is Noetherian module . Then N is finitely generated . As M is finitely presented , so N is direct summand submodule by [6,Lam.Exc.32,P.163]. Then $N \oplus W = M$ for some a submodule W of M , so $M/N \cong W$. Consider $(i \circ f)(M/N) + N = W \oplus N = M$. Then M is strongly P-coretractable module .

By a similar proof Corollary(1.6) , Proposition(1.14) and Theorem(1.18) , we can get the following result .

Corollary*(3.11)*:If M is a purely lifting V-module. Then M is a strongly P-coretractable if and only if M is a strongly coretractable module .

Proposition*(3.12):* Let M be a Noetherian projective R-module . Then M is a strongly P-coretractable module .

Theorem*(3.13)*: Let $\{ M_\alpha : \alpha \in I \}$ be a family of strongly P-coretractable R-module if for any $\alpha, \beta \in I$, M_α is M_β–injective and $M = \bigoplus_{\alpha \in I} M_\alpha$ has PIP , then M is a strongly P-coretractable . In particular , if M is quasi-injective P-coretractable and satisfy PIP , then M is P-coretractable for any index I .

Proposition*(3.14):* Let M be a quasi-Dedekind R-module , then the following statements are equivalent :

(1) M is a strongly P-coretractable ;
(2) M is a P-coretractable ;
(3) M is a pure simple ;
(4) M is a mono-P-coretractable .

Proof : (1)⇒(2) It is clear by Examples and Remarks(3.2(5)) .
(2)⇒(3) It follows by Proposition(1.7) since M is a quasi-Dedekind module.
(3)⇒(4) It follows by Examples and Remarks(2.2 (3)) .

(4)⇒(1) Let M be a mono-P-coretractable . It is clear that M is P-coretractable. As M is quasi-Dedekind , so again M is a pure simple by Proposition(1.7) and hence M is strongly P-coretractable by Examples and Remarks(3.2(3)) .

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