Y-coretractable and Strongly Y-coretractable Modules

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ABSTRACT--- ***Let R be a ring with identity and M be an R-module with unitary . In this paper we introduce the notion of Y-coretractable module . Some basic properties of this class of modules are investigated and some relationships between these modules and other related concepts are introduced .***

Keywords--- coretractable module , Y-coretractable module , C-coretractable module

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1. INTRODUCTION

Throughout this paper all rings have identities and all modules are unital right R-modules . A module M is called coretractable if for each a proper submodule N of M , there exists a nonzero R-homomorphism $f:M/N\to M$ [1] , and an R-module M is called strongly coretractable module if for each proper submodule N of M , there exists a nonzero R-homomorphism $f:M/N\to M$ such that Imf+N=M [2]. It is clear every strongly coretractable module is coretractable but it is not conversely [2]. In this paper, we introduce the notion of Y-coretractable module where an R-module M is called Y-coretractable if for each proper y-closed submodule N of M , there exists a nonzero homomorphism $f\in Hom_R(M/N,M)$. It is clear every coretractable module is Y-coretractable but it is not conversely. In this work we study some basic properties of this notion and give the notion of strongly Y-coretractable and mono-Y-coretractable modules . A characterizations of these modules are given . Some examples and remarks are given and some important characterizations and properties are investigated . Also we study its relation with other concepts . We recall the following concepts a submodule N of M is called y-closed submodule if M/N is nonsingular module . And every y-closed submodule is closed but not conversely, for example $<\bar{2}>$ in Z_6 is closed , not y-closed submodule [8,P.19] . If a module is nonsingular , then the concepts closed and y-closed are coincide [3] . Note that asgari and Haghany in[4], introduced the concept of t-closed submodule . However the two concepts t-closed and y-closed submodule are coincide [see Proposition(2.6)] in [5]. An R-module M is called CLS-module if every y-closed submodule is direct summand submodule [12] . Also asgari and Haghany in [5] , introduced the concept of t-extending , where a module is t-extending if every t-closed submodule is direct summand . Thus t-extending and CLS-module are coincide too . However M is a nonsingular R-module . Then M is extending module if and only if M is CLS-module [12] . End(M) is the set of endomorphisms of M .

2. Y- CORETRACTABLE MODULES

Definition(1.1): An R-module is called Y-coretractable module if for each y-closed proper submodule N of M , there exists a nonzero homomorphism $f\in Hom(M/N, M)$.

Equivalently , M is an Y-coretractable R-module if for each proper y-closed submodule N of M , there exists $g\in End_R(M)$, $g\neq 0$ and $g(N)=0$. A ring R is called Y-coretractable if R is Y-coretractable R-module .

Examples and Remarks(1.2)

(1) It is clear that every coretractable module is Y-coretractable . But the converse is not true , for example Z as Z-module is Y-coretractable , but it is not coretractable .

(2) Y-coretractability is not preserved by taking submodules , factor modules and direct summands , since for any R-module M and a cogenerator R-module C , $C\oplus M$ is a cogenerator by [1], and so $C\oplus M$ is a coretractable module .

Thus C⊕M is Y-coretractable , but M need not be Y-coretractable . Where an R-module M is called cogenerator if for every nonzero homomorphism $f:M_1\longrightarrow M_2$ where M_1 and M_2 are R-modules , there exists $g:M_2\longrightarrow M$ such that $g\circ f\neq 0$.

(3) Every C-coretractable R-module is Y-coretractable module . Where an R-module M is called C-coretractable module if for each proper closed submodule N of M , there exists a nonzero homomorphism $f\in Hom_R(M/N,M)$ [3].

Proof : It is clear , since every y-closed submodule is closed .

(4) Every semisimple module is Y-coretractable . For example $Z_2\oplus Z_2$, Z_6 , … are semisimple module and hence Y-coretractable , but the converse is not true for example the Z-module $Z_2\oplus Z_8$ is C-coretractable , and so Y-coretractable , but it is not semisimple .

(5) Every CLS-module is an Y-coretractable module .

Proof : Since M is CLS-module , so every y-closed submodule is a direct summand and hence it is clear that M is Y-coretractable module .

For example , the Z-module $M=Q\oplus Z_2$ is CLS-module , hence M is Y-coretractable module .

(6) Every singular R-module is Y-coretractable module .

Proof: Since every singular module is CLS-module by [8,Remark(2.3.3]. Hence by Examples and Remarks(1.2(5)) , then the result is hold ◙

For example $M= Z_4\oplus Z_2$ as Z-module is singular , hence it is Y-coretractable .

(7) Let M≅M' where M is an Y-coretractable R-module . Then M' is Y-coretractable .

Proof : Since M≅M' , so there exists $f:M\longrightarrow M'$ be R-isomorphism . Let W be a proper y-closed submodule of M' . Then $N=f^{-1}(W)$ is proper y-closed submodule of M . Since M is Y-coretractable module . So there exists a nonzero R-homomorphism $h:M/N\longrightarrow M$ such that $h(M/N)+N=M$. Now, the mapping $f\circ g\circ f^{-1}\in End_R(M')$. Moreover, $f\circ g\circ f^{-1}(M')=f\circ g(f^{-1}(M'))= f\circ g(M)=f(g(M))$. But $g(M)\neq 0$ and f is an isomorphism , so $f(g(M))\neq 0$. Thus $f\circ g\circ f^{-1}(M')\neq 0$. Also $f\circ g\circ f^{-1}(W) = f\circ g(f^{-1}(W)) = f\circ g(N)= f(g(N)) = f(0) =0$. Thus M' is Y-coretractable .

(8) Every extending module M is an Y-coretractable .

Proof : Let M be an extending module . Thus M is a C-coretractable by [3] and hence it is Y-coretractable ◙

We can prove the following two results .

Proposition(1.3): A fully invariant direct summand N of Y-coretractable R-module M is a Y-coretractable module .

Proof : Since N is a direct summand of M , so there exists a submodule W of M such that N⊕W=M . Let K be a proper closed submodule of N , then K⊕W is a closed submodule in N⊕W=M . Since M is a Y-coretractable module , so there exists $f\in End_R(M)$, $f\neq 0$ and $f(K\oplus W)=0$. Suppose that g is a restriction map from N into M , $g\neq 0$. Also N is fully invariant direct submodule . Then N is stable module [5,Lemma(2.1.6)] . So $g(N)\subseteq N$. Therefore $g\in End_R(N)$, $g\neq 0$. $g(K)=f|_N(K)=0$. Thus N is a Y-coretractable ◙

Corollary(1.4): Let N be a direct summand submodule of duo R-module M . If M is a Y-coretractable . Then N is also Y-coretractable .

Recall that an R-module M is called a *generalized extending* module if for any submodule N of M , there exists a direct summand K of M such that N⊆K and K/N is singular [6] , and An R-module M is called *Y-extending* if for every submodule N of M , there exists a direct summand D of M such that $N\cap D\leq_e N$ and $N\cap D\leq_e D$ [7] .

Proposition(1.5): Let M be an R-module . If M satisfies any one of the following conditions :

(1) M is a generalized extending module ;

(2) M is an Y-extending module .

Then M is an Y-coretractable module

Proof : If condition(1) hold . Since every generalized extending module is CLS-module by [8,Proposition(2.3.13)] . Thus M is Y- coretractable . If condition(2) hold . Since every Y-extending module is CLS-module by [8, Proposition(2.3.14) , P.44] . Hence M is Y-coretractable , by Examples and Remarks (1.2(5)) ◙

Proposition(1.6): Let M be a nonsingular R-module .Then M is Y-coretractable module if and only if M is C-coretractable module .

Proof : It is clear , since the two concepts closed and y-closed submodules are coincide in the class of nonsingular modules ◙

Proposition(1.7): Let M , A and B be R-modules . Consider the short exact $0 \to A \xrightarrow{f} B \xrightarrow{g} M \to 0$. If f(A) is an essential submodule of B , then M is Y-coretractable module .

Proof : Since f(A) is an essential submodule of B . Hence B/f(A) is singular module by [9, page32] . But g is an epimorphism which implies B/kerg ≅ M . But exactness of the sequence kerg=f(A). Hence B/f(A) ≅M . Therefore M is singular module , and so by Examples and Remarks (1.2(6)) , M is Y-coretractable module ◙

Corollary(1.8): Let M be an R-module . Then M/N is Y-coretractable module for each proper essential submodule N of M .

Proof : Suppose that N be a proper essential submodule of M . Consider the short exact $0 \to N \xrightarrow{i} M \xrightarrow{\pi} M/N \to 0$ where π is the natural epimorphism from M into M/N . Then by Proposition(1.7) , we get M/N is Y-coretractable module .

We get the following immediately by Corollary(1.8)

Corollary(1.9): Let M be an R-module. If M is a uniform module . Then M/N is an Y-coretractable module for each submodule N of M .

As a consequence of Corollary(1.8) and Corollary(1.9) , the Z-module Q/Z is Y-coretractable .

Proposition(1.10): Let M be a projective R-module . If $A \leq^{\oplus} E(M)$ such that A∩M is y-closed submodule of M .Then M is an Y-coretractable module .

Proof : The hypothesis implies M is a CLS-module by [8,Proposition(2.4.3)] . Hence by Examples and Remarks(1.2(5)), M is an Y-coretractable module

Proposition(1.11): Let M be an R-module . If M is Y-coretractable and for any proper submodule N of M , there exists y-closed submodule W of M such that N⊂W . Then M is a coretractable .

Proof : Let N be a proper submodule . By hypothesis there exists y-closed submodule W of M such that N⊂W . But M is Y-coretractable , hence there exists $f \in End_R(M)$, f≠0 and f(W)=0 which implies that f(N)=0 . Therefore M is coretractable module

We can get easily the following two results by similar to [10,Theorem(1.11)] and [10,Theorem(1.13)] , respectively .

Theorem(1.12): Let $M = \bigoplus_{\alpha \in I} M_\alpha$, where M_α is an R-module for all α∈I such that every y-closed submodule of M is fully invariant submodule. If M_α is Y-coretractable module for all α∈I , then M is a Y-coretractable .

Theorem(1.13): Let M_1 and M_2 be R-modules , $M=M_1 \oplus M_2$ such that $ann(M_1)+ann(M_2)=R$. If M_1 and M_2 are Y-coretractable module , then M is a Y-coretractable module .

Proposition(1.14): Every quasi-Dedekind Y-coretractable R-module is nonsingular .

Proof : Let N be a proper y-closed submodule of M . Then M/N is nonsingular . As M is Y-coretractable , there exists $f \in End_R(M)$, f≠0 and N ⊆ kerf . But kerf=0 (since M is quasi-Dedekind) , so N=0 . It follows that M/(0)≅M , so M is nonsingular module .

3. MONO-Y-CORETRACTABLE MODULES

Before we introduce this notion we need to show two concepts , an R-module M is called mono-C-coretractable if for each proper closed submodule of M , there exists $f\in End_R(M)$, $f\neq 0$ and N=kerf [4] and M is called mono-coretractable if for each proper submodule of M , there exists $f\in End_R(M)$, $f\neq 0$ and N=kerf . Clearly that every mono-coretractable module is mono-C-coretractable but not conversely [4] .

Definition(2.1): An R-module M is called *mono-Y-coretractable* if for each proper y-closed submodule N of M , there exists $f\in End_R(M)$, $f\neq 0$ and N=kerf .

Examples and Remarks(2.2):

(1) Every mono-coretractable is mono-Y-coretractable but not conversely , for example , Z as Z-module is mono-Y-coretractable , but not mono-coretractable .

(2) Every mono-C-coretractable module implies mono-Y-coretractable.

(3) Every semisimple module is mono-Y-coretractable .

Definition(2.3): An R-module M is called Y-fully stable if for each y-closed submodule is stable .

It is clear every fully stable is Y-fully stable , but not conversely . For example , $M=Z_8\oplus Z_2$ as Z-module . The only y-closed submodule is M and M is stable , so M is Y-fuuly stable . But $f:N\to M$, where $N=<(\bar{2},\bar{1})>=\{(\bar{0},\bar{0}),(\bar{2},\bar{1}),(\bar{4},\bar{0}),(\bar{6},\bar{1})\}$, define by $f(((\bar{x},\bar{y}))=(\bar{x},\bar{0})$ for all $(\bar{x},\bar{y})\in N$. It is clear that $f(N)\not\subseteq N$, so M is not fully stable

Proposition(2.4): Let M be a strongly Rickart R-module . Then M is mono-Y-coretractable if and only if M is Y-fully stable CLS-module .

Proof : (⟹) Let N be a proper y-closed submodule of M . Since M is mono-Y-coretractable module , so there exists $f\in End_R(M)$, $f\neq 0$ and kerf=N . But M is strongly Rickart , so kerf is stable direct summand . Thus M is Y-fully stable and M is CLS-module .

(⟸) It is clear .

Now , we can prove the following Propositions by similar proof of [10,Proposition(3.6)] .

Proposition(2.5): Let M be a mono-Y-coretractable Y-fully stable R-module . Then every nonzero y-closed submodule of M is also mono-Y-coretractable module .

Proposition(2.6): Let $M=M_1\oplus M_2$ where M_1 and M_2 be R-modules such that every y-closed submodule of M is fully invariant . If M_1 and M_2 are mono-Y-coretractable modules , then M is mono-Y-coretractable .

Proof : Let K be a proper y-closed submodule of M . Since K is fully invariant by hypothesis, $K=K_1\oplus K_2$ where K_1 is y-closed submodule of M_1 and K_2 is y-closed submodule of M_2 .

Case(1): K_1 is a proper submodule of M_1 and $K_2=M_2$. Since M_1 is mono-Y-coretractable module , so there exists $f\in End_R(M_1)$ $f\neq 0$ and $kerf=K_1$. Define $g:M\longrightarrow M$ by $g((x,y))=(f(x),0)$ for each $(x,y)\in M$. It is clear that $g\in End_R(M)$, $g\neq 0$ and kerg=K .

Case(2): If $K_1=M_1$ and K_2 is a proper submodule of M_2, then by a similar proof of case(1) , we can get the result .

Case(3): K_1 is a proper submodule of M_1 and K_2 is a proper submodule of M_2 . Since M_1 and M_2 are mono-Y-coretractable module , then there exist $f\in End_R(M)$, $g\in End_R(M)$ such that $f\neq 0$ and $g\neq 0$. Define $h\in End_R(M)$ by $h((x,y))=(f(x),g(y))$ for each $(x,y)\in M$. Then $h\in End_R(M)$, $h\neq 0$ and kerh=K .

Now , we shall present another generalization for the concept strongly cortractable as the following :

4. STRONGLY Y- CORETRACTABLE MODULES

Definition(3.1): An R-module M is called *strongly Y-coretractable* module if for each proper y-closed submodule N of M , there exists a nonzero homomorphism $f\in Hom_R(M/N,M)$ and Imf +N =M .

Equivalently , M is strongly Y-coretractable R-module if for each proper y-closed submodule N of M , there exists $g\in End_R(M)$, $g\neq 0$, g(N)=0 and Imf+N=M . A ring R is called strongly Y-coretractable if R is strongly Y-coretractable R-module .

Examples and Remarks(3.2):

(1) Every semisimple module is a strongly Y-coretractable .

(2) Every CLS-module is a strongly Y-coretractable . For example , Z_4 as Z_4-module has no proper y-closed , so Z_4 is CLS-module and hence strongly Y-coretractable module .

(3) Every singular R-module is a strongly Y-coretractable .

(4) Every strongly coretractable is a strongly Y-coretractable . But the converse is not true in general . For example consider $M=Z_{12}$ as Z-module is not strongly coretractable module by part(1) , but M is singular module , and hence M is a strongly Y-coretractable by part(3).

(5) Every strongly C-coretractable R-module is strongly Y-coretractable.

Proof : It is clear since every y-closed submodule is closed . ◙

(6) If M be a faithful multiplication module over CLS-ring, then M is a strongly Y-coretractable module .

Proof : By [8, Proposition(2.3.12),P.43] , M is CLS-module . Hence by part (2) , M is strongly Y-coretractable .

(7) Let M be a nonsingular R-module . Then M is strongly Y-coretractable module if and only if M is strongly C-coretractable .

Proof : It is clear since y-closed and closed submodule are coinside under nonsingular module .

Proposition(3.3): Let M≅M' where M is a strongly Y-coretractable R-module . Then M' is strongly Y-coretractable module

Proof : Since M≅M' , so there exists f:M⟶M' be R-isomorphism . Let W be a proper y-closed submodule of M' . Then $N=f^{-1}(W)$ is a proper y-closed submodule of M . Since M is strongly Y-coretractable module . So there exists a nonzero R-homomorphism h:M/N ⟶ M such that h(M/N)+N=M . And we get easily that M' is a strongly Y-coretractable R-module ◙

Proposition(3.4): Let M be a strongly Y-coretractable R-module and N be a proper Y-closed submodule of M ,then M/N is strongly Y-coretractable module .

Proof : Let W/N be a proper y-closed submodule of M/N . Since N is y-closed submodule of M , so W is y-closed submodule of M. But M is strongly Y-coretractable module , hence there exists a nonzero R-homomorphism g:M/W→M and g(M/W)+W=M . But

(M/N)/(W/N) ≅M/W . Set $f=\pi^{o}g$ where π is the natural epimorphism from M into M/W . Then $f(\frac{M}{W})+\frac{W}{N}=\pi(g(\frac{M}{W}))+\frac{W}{N}=\frac{g(\frac{M}{W})+N}{N}+\frac{W}{N}=\frac{g(\frac{M}{W})+N+W}{N}=\frac{M+N}{N}=\frac{M}{N}$.

we can get M/N is also strongly Y-coretractable module . ◙

Proposition(3.5): If M is CLS-module , then M/N is a strongly Y- coretractable module for each N≤M .

Proof : Let N be a proper submodule of M . Since M is CLS-module , hence M/N is CLS-module by [8, Proposition(2.3.11) ,P.42] and hence M/N is strongly Y-coretractable module by Examples and Remarks(3.2 (2)) ◙

Recall that , let M be an R-module . The second singular submodule $Z_2(M)$ is a submodule in M containing Z(M) such that $Z_2(M)/Z(M)$ is singular submodule of M/ Z(M) [8] . Consider Z-module Q. $Z_2(Q)=0$. Consider Z-module Z_n . $Z_2(Z_n)=Z_n$.

Proposition(3.6): Let M be an R-module , then $M/Z_2(M)$ is a strongly Y- coretractable module .

Proof: Since $Z_2(M)$ is y-closed submodule of M by [8,Remark(2.2.3), P.32] , then $M/Z_2(M)$ is CLS-module by [8, Proposition(2.3.11) , P.42] , and hence it is strongly Y-coretractable module by Examples and Remarks(3.2 (2)) .

Proposition(3.7): Let $M=M_1\oplus M_2$, where M is duo module (or distributive or $annM_1+annM_2 = R$) . Then M is strongly Y-coretractable module , if M_1 and M_2 are strongly Y-coretractable module .

Proof : Let N be a proper y-closed submodule of M . Since M is duo or distributive , then $N=(N\cap M_1)\oplus(N\cap M_2)$. Thus $N=N_1\oplus N_2$ for some $N_1\leq M_1$ and $N_2\leq M_2$. Thus each of N_1 and N_2 are y-closed submodules M_1 and M_2 respectively.

Case(1): K_1 is a proper submodule of M_1 and K_2 is a proper submodule of M_2 . Since M_1 and M_2 are strongly coretractable module , then there exists $f : M_1/K_1 \longrightarrow M_1$ such that $Imf+K_1=M_1$, and there exists $g:M_2/K_2\longrightarrow M_2$ such that $Img+K_2=M_2$.

Now, define a nonzero R-homomorphism $h:M/K\to M$;

that is $h:(M_1\oplus M_2)/(K_1\oplus K_2)\longrightarrow M_1\oplus M_2$ by $h[(m_1,m_2)+K_1\oplus K_2]=[f(m_1\oplus K_1) , g(m_2\oplus K_2)]$. Then h is well-defined . Now , $Imh+K=Imh +(K_1\oplus K_2)=(Imf\oplus Img)+ (K_1\oplus K_2) =(Imf+K_1)\oplus (Img+K_2) =M_1\oplus M_2=M$.

Case(2): $K_1=M_1$ and K_2 is a nonzero proper submodule of M_2 .

Consider $(M_1\oplus M_2)/(K_1\oplus K_2)=(M_1\oplus M_2)/(M_1\oplus K_2)\cong M_2/K_2$. Since M_2 is strongly coretractable module , so there exists $f:M_2/K_2\longrightarrow M_2$ such that $Imf + K_2 = M_2$. Define $g : M_2/ K_2 \longrightarrow M_1 \oplus M_2$ by $g = i\circ f$ where i is the inclusion mapping from M_2 into $M_1\oplus M_2$. Therefore $Img+K= Img+ (K_1\oplus K_2) = Img+(M_1\oplus K_2) = M_1\oplus(Img+K_2) = M_1\oplus((0)\oplus Imf)+K_2) = M_1\oplus((0)+(Imf+K_2) = M_1\oplus((0)+M_2) =M_1\oplus M_2 = M$.

Case(3): K_1 is a nonzero proper submodule of M_1 and $K_2 = M_2$, then by a similar proof case(2) , we can get the result

Case(4): $K_1=0$ and $K_2 = M_2$.

Consider $M/K\cong (M_1\oplus M_2)/(0\oplus K_2) \cong M_1$. Let i is the inclusion mapping from M_1 into $M_1\oplus M_2$, hence $i(M_1)+K=(M_1 \oplus 0) \oplus(0\oplus M_2)=M$

Case(5): $K_1=M_1$ and $K_2=0$, then the proof is similar to Case(4) .

M is a strongly Y-coretractable module . ◙

Now, we can get the following result directly by Proposition(1.5) and Remark and Example(3.2 (2)) :

Proposition(3.8): Let M be an R-module . Then M is strongly Y-coretractable module if M satisfies any one of the following conditions :

(1) M is generalized extending module ;

(2) M is Y-extending module ;

Also , we can get the following result by a similar proof of Proposition(1.7) :

Proposition(3.9): Let M,A and B be R-modules with the short exact $0 \to A \xrightarrow{f} B \xrightarrow{g} M\to 0$. If f(A) is an essential submodule of B , then M is strongly Y-coretractable module .

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