The Bi-Ideals in left almost Rings

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ABSTRACT— ***In this study we promoted some notion of a left almost semiring defined in (Mrudula Devi and Sobha Latha, 2015) and further established the substructures, operations on substructures of left almost semirings. We also indicated the non similarity of a left almost semiring to the usual notion of a semiring.***

Keywords— left almost semiring, bi-ideal, quasi-ideal, left ideal, ideal.

1. INTRODUCTION

A groupoid S is called a left almost semigroup, abbreviated as an LA-semigroup, if its elements satisfy the left invertive law [2, 3], that is: $(ab)c = (cb)a$ for all $a, b \in S$ several examples and interesting properties of LA-semigroups can be found in [5, 6, 7, 8]. It has been shown in [5] that if an LA-semigroup contains a left identity then it is unique. It has been proved also that an LA-semigroup with right identity is a commutative monoid, that is, a semigroup with an identity element. It is also known [2] that in an LA-semigroup, the medial law, that is, $(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in S$ holds. Now we define the concepts that we will used. Let S be an LA-semigroup. By an LA-subsemigroup of [9], we mean a non-empty subset A of S such that $A^2 \subseteq A$.

Yusuf in [13] introduced the concept of a left almost ring (LA-ring). That is, a non-empty set R with two binary operations "+" and "$\cdot$" is called a left almost ring, if $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semigroup and distributive laws of "$\cdot$" over "+" hold. Further in [10] Shah and Rehman generalized the notions of commutative rings into LA-rings. However Shah and Fazal ur Rehman in [10] generalize the notion of an LA-ring into a near left almost ring. A near left almost ring (nLA-ring) N is an LA-group under "+", an LA-semigroup under "$\cdot$" and left distributive property of "$\cdot$" over "+" holds.

Shah, Fazal ur Rehman and Raees asserted that a commutative ring $(R, +, \cdot)$ we can always obtain an LA-ring $(R, \oplus, \cdot)$ by defining, for $a, b \in R, a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. Furthermore, in this paper we characterize the bi-ideal and quasi-ideal ideals in left almost semirings.

2. BASIC PROPERTIES

In this section, we refer to the article of Mrudula Devi and Sobha Latha in 2015, for some elementary aspects and quote few definitions, and essential examples to step up this study. For more details, we refer to the papers in the references.

Definition 2.1. [1] A left almost semiring is a triple $(R, +, \cdot)$ of a nonempty set R together with two binary operations "+" and "$\cdot$" (called addition and multiplication respectively) defined on R such that the followings hold:

1. $(R, +)$ is a left almost semigroup.
2. $(R, \cdot)$ is a left almost semigroup.
3. $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$, for all $a, b, c \in R$.

If R contains an element 0 such that $0 + x = x$ and $0x = 0$ for all $x \in R$, then 0 is called the left zero element of the left almost semiring R. Throughout this paper, R will always denote a left almost semiring with a left zero and unless otherwise stated a left almost semiring means a left almost semiring with left zero.

Definition 2.2. A left almost semiring R with a left identity $e,$ such that $e{\cdot}a = a$ for all $a \in R,$ is called a left almost semiring with a left identity.

Proposition 2.3. Let R be a left almost semiring with left identity. Then $RR = R$ and $R = eR = Re.$

Lemma 2.4. [4] Let R be a left almost semiring. Then $(ab)(cd) = (ac)(bd),$ for all $a,b,c \in R.$

Definition 2.5. [4] A nonempty subset S of a left almost semiring R is said to be a left almost subsemiring if and only if S is itself a left almost semiring under the same binary operations as in $R.$

Lemma 2.6. A non empty subset S of a left almost semiring R is a left almost semiring if and only if $a+b \in S$ and $a{\cdot}b \in S$ for all $a,b \in S.$

Definition 2.7. [4] A left almost subsemiring I of a left almost semiring R is called a left ideal of R if $RI \subseteq I,$ and I is called a right ideal of R if $IR \subseteq I$ and is called two sided ideal or simply ideal of R if it is both left and right ideal of R .

3. QUASI-IDEALS IN LEFT ALMOST SEMIGROUPS

The results of the following lemmas seem to play an important role to study left almost semirings; these facts will be used frequently and normally we shall make no reference to this lemma.

Lemma 3.1. If R is a left almost semiring with a left identity, then $\langle a \rangle = aR$ is a left ideal of $R,$ where $a \in R.$

Proof. Let $t \in R$ and $ar, as \in \langle a \rangle.$ Then $ar + ar = a(r+s) \in aR = \langle a \rangle$ and $t(ar) = a(tr) \in aR = \langle a \rangle$ so $\langle \mathrm{a} \rangle$ is a left ideal of $R.$

Proposition 3.2. Let R be a left almost semiring with a left identity and let A be a left almost subsemiring of $R.$ Then $(A:R)$ is a left almost subsemiring in $R,$ where $(A:R) = \{r \in R : Rr \subseteq A\}.$

Proof. Let R be a left almost semiring and let A be a left almost subsemiring of $R.$ Suppose that $t \in R$ and $r,s \in (A:R).$ Then $Rr \subseteq A$ and $Rs \subseteq A$ so that

$$\begin{aligned} R(r+s) &= Rr + Rs \\ &\subseteq A + A \\ &= A \end{aligned}$$

and

$$\begin{aligned} R(rs) &= r(Rs) \\ &\subseteq rA \\ &= A. \end{aligned}$$

Then $(A:R)$ is a left almost subsemiring in $R.$

Lemma 3.3. Let R be a left almost semiring and $\varnothing \neq X \subseteq R.$ If $\langle X \rangle = \left\{ \sum_{i=1}^{n} x_i : x_i \in X \right\},$ then $\langle X \rangle$ is a left almost subsemigroup of $(R,+).$

Proof. Let R be a left almost semiring and $x,y \in \langle X \rangle.$ Then $x = \sum_{i=1}^{n} x_i$ and $y = \sum_{i=1}^{m} y_i$ where $x_i, y_i \in X.$ If $n \geq m,$ then

$$
\begin{aligned}
x+y &= \sum_{i=1}^{n} x_i + \sum_{i=1}^{m} y_i \\
&= \left(\left(\left(x_1+x_2\right)+x_3\right)+\ldots+x_n\right)+\left(\left(\left(y_1+y_2\right)+y_3\right)+\ldots+y_m\right) \\
&= \left(\left(\left(x_1+x_2\right)+x_3\right)+\ldots+x_n\right)+\left(\left(\left(\left(\left(0+0\right)+\ldots+0\right)+y_1\right)+y_2\right)+\ldots+y_m\right) \\
&= \left(\left(\left(\left(x_1+x_2\right)+x_3\right)+\ldots+x_{n-1}\right)+\left(\left(\left(\left(0+0\right)+\ldots+0\right)+y_1\right)+\ldots+y_{m-1}\right)\right)+\left(x_n+y_m\right) \\
&= \left(\left(\left(\left(x_1+x_2\right)+x_3\right)+\left(\left(0+0\right)+0\right)\right)+\ldots+\left(x_{n-1}+y_{m-1}\right)\right)+\left(x_n+y_n\right) \\
&= \left(\left(\left(\left(x_1+x_2\right)+\left(0+0\right)\right)+\left(x_3+0\right)\right)+\ldots+\left(x_{n-1}+y_{m-1}\right)\right)+\left(x_n+y_m\right) \\
&= \left(\left(\left(\left(x_1+0\right)+\left(x_2+0\right)\right)+\left(x_3+0\right)\right)+\ldots+\left(x_{n-1}+y_{m-1}\right)\right)+\left(x_n+y_m\right) \\
&= \sum_{i=1}^{n} \left(x_i+y\right)_i \in \langle X \rangle \\
&= \sum_{i=1}^{\max\{n,m\}} \left(x_i+y\right)_i \in \langle X \rangle.
\end{aligned}
$$

Hence $\langle X \rangle$ is a left almost subsemigroup of $(R,+)$.

Proposition 3.4. Let R be a left almost semiring with left identity and $\varnothing \neq X, Y, Z \subseteq R$. Then

1. $X \subseteq \langle X \rangle$,
2. $XY \subseteq \langle XY \rangle$,
3. if $X \subseteq Y$, then $\langle X \rangle \subseteq \langle Y \rangle$,
4. $\langle X \cup Y \rangle = \langle \langle X \rangle \cup \langle Y \rangle \rangle$,
5. $\langle XY \rangle = \langle \langle X \rangle Y \rangle = \langle X \langle Y \rangle \rangle = \langle \langle X \rangle \langle Y \rangle \rangle$,
6. $\langle (XY)Z \rangle = \langle \langle XY \rangle Z \rangle = \langle \langle ZY \rangle X \rangle$,
7. $\langle X \cap Y \rangle = \langle X \rangle \cap \langle Y \rangle$,
8. $\langle X \rangle \langle Y \rangle \subseteq \langle XY \rangle$.
9. $\langle R \rangle = R$.

Proof. 1. Let $x \in X$. Then $x = \sum_{i=1}^{1} x_i \in \langle X \rangle$, where $x_1 = x$ so that $X \subseteq \langle X \rangle$.

2. This follows from 1.

3. Suppose that $X \subseteq Y$. Let $x \in \langle X \rangle$. Then $x = \sum_{i=1}^{n} x_i$, where $x_i \in X \subseteq Y$, so that $x = \sum_{i=1}^{n} x_i \subseteq \langle Y \rangle$.

4. By 1, we have $X \subseteq \langle X \rangle$ and $Y \subseteq \langle Y \rangle$. Then $X \cup Y \subseteq \langle X \rangle \cup \langle Y \rangle$ so that $\langle X \cup Y \rangle \subseteq \langle \langle X \rangle \cup \langle Y \rangle \rangle$. Conversely, let $x \in \langle \langle X \rangle \cup \langle Y \rangle \rangle$. Then $x = \sum_{i=1}^{n} x_i$ where $x_i \in \langle X \rangle \cup \langle Y \rangle$ so that $x_i \in \langle X \rangle$ or $x_i \in \langle Y \rangle$. Therefore $x_i = \sum_{i=1}^{m} a_i$ or $x_i = \sum_{i=1}^{k} b_i$ where $a_i \in X, b_i \in Y$. This implies that x is a finite sum of elements from X or Y. Hence $\langle \langle X \rangle \cup \langle Y \rangle \rangle \subseteq \langle X \cup Y \rangle$ so $\langle X \cup Y \rangle = \langle \langle X \rangle \cup \langle Y \rangle \rangle$.

5. By 1, we have $X \subseteq \langle X \rangle$. Then $XY \subseteq \langle X \rangle Y$ so that $\langle XY \rangle \subseteq \langle \langle X \rangle Y \rangle$. Conversely, let $x \in \langle \langle X \rangle Y \rangle$. Then $x = \sum_{i=1}^{n} z_i$ where $z_i \in \langle X \rangle Y$ so that $z_i = \left(\sum_{i=1}^{m} a_i \right) b = \sum_{i=1}^{m} a_i b$. This implies that x is a finite sum of elements from XY. Hence $\langle \langle X \rangle Y \rangle \subseteq \langle XY \rangle$ so $\langle XY \rangle = \langle \langle X \rangle Y \rangle$. Similarly, $\langle XY \rangle = \langle \langle X \rangle Y \rangle = \langle X \langle Y \rangle \rangle = \langle \langle X \rangle \langle Y \rangle \rangle$.

6. By 5, we get $\langle (XY)Z \rangle = \langle \langle XY \rangle Z \rangle = \langle Z \langle YX \rangle \rangle$.

7. Since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$, we get $\langle X \cap Y \rangle \subseteq \langle X \rangle$ and $\langle X \cap Y \rangle \subseteq \langle Y \rangle$ so that $\langle X \cap Y \rangle \subseteq \langle X \rangle \cap \langle Y \rangle$. Conversely, let $x \in \langle X \rangle \cap \langle Y \rangle$. Then $x \in \langle X \rangle$ and $x \in \langle Y \rangle$ so $x = \sum_{i=1}^{n} x_i$ and $x = \sum_{i=1}^{m} y_i$ where $x_i \in X, y_i \in Y$. Therefore $\sum_{i=1}^{n} x_i = \sum_{i=1}^{m} y_i$ where $x_i \in X, y_i \in Y$. This implies that x is a finite sum of elements from X and Y. Hence $\langle X \rangle \cap \langle Y \rangle \subseteq \langle X \cap Y \rangle$ so $\langle X \rangle \cap \langle Y \rangle = \langle X \cap Y \rangle$.

8. Let $x \in \langle X \rangle \langle Y \rangle$. Then

$$
\begin{aligned}
x &= \sum_{i=1}^{n} x_i \sum_{i=1}^{m} y_i \\
&= \left(\left((x_1 + x_2) + x_3 \right) + \ldots + x_n \right) \sum_{i=1}^{m} y_i \\
&= \left(\left(x_1 \sum_{i=1}^{m} y_i + x_2 \sum_{i=1}^{m} y_i \right) + x_3 \sum_{i=1}^{m} y_i \right) + \ldots + x_n \sum_{i=1}^{m} y_i \\
&= \left(\left(\sum_{i=1}^{m} x_1 y_i + \sum_{i=1}^{m} x_2 y_i \right) + \sum_{i=1}^{m} x_3 y_i \right) + \ldots + \sum_{i=1}^{m} x_n y_i \\
&= \left(\left(\sum_{i=1}^{m} x_1 y_i + \sum_{i=1}^{m} x_2 y_i \right) + \sum_{i=1}^{m} x_3 y_i \right) + \ldots + \sum_{i=1}^{m} x_n y_i \\
&\in \left(\left(\langle x_1 Y \rangle + \langle x_2 Y \rangle \right) + \langle x_3 Y \rangle \right) + \ldots + \langle x_n Y \rangle \\
&\subseteq \left(\left(\langle XY \rangle + \langle XY \rangle \right) + \langle XY \rangle \right) + \ldots + \langle XY \rangle \\
&= \langle XY \rangle
\end{aligned}
$$

where $x_i \in X, y_i \in Y$. Hence $\langle X \rangle \langle Y \rangle \subseteq \langle XY \rangle$.

9. Clearly, $\langle R \rangle = R$.

Theorem 3.5. Let R be a left almost semiring with left identity and A, B be LA-subsemigroups of $(R, +)$. Then

1. A is a left ideal of R if and only if $\langle RA \rangle \subseteq A$,
2. B is an ideal of R if and only if $\langle BR \rangle \subseteq B$.

Proof. 1. Suppose that A is a left ideal of R. Let $x \in \langle AR \rangle$. Then $x = \sum_{i=1}^{n} a_i r_i$ where $a_i \in A, r_i \in R$. But $RA \subseteq A$, we get so that $x = \sum_{i=1}^{n} x_i r_i \in A$. Hence $\langle RA \rangle \subseteq A$. On the contrary, suppose that $\langle RA \rangle \subseteq A$. Then by Proposition 3.4, we get $RA \subseteq \langle RA \rangle \subseteq A$. Hence A is a left ideal of R.

2. Suppose that B is an ideal of $R.$ Let $x \in \langle BR \rangle.$ Then $x = \sum_{i=1}^{n} r_i b_i$ where $b_i \in B, r_i \in R.$ But $BR \subseteq B,$ we get so that $x = \sum_{i=1}^{n} r_i b_i \in B.$ Hence $\langle BR \rangle \subseteq B.$ On the contrary, suppose that $\langle BR \rangle \subseteq B.$ Then by Proposition 3.4, we get $BR \subseteq \langle BR \rangle \subseteq B.$ Hence A is an ideal of $R.$

Definition 3.6. Let R be a left almost semiring. A left almost subsemiring Q is called a quasi-ideal of R if $QR \cap RQ \subseteq Q.$

Proposition 3.7. Let R be a left almost semiring with a left identity and $\varnothing \neq X \subseteq R.$ Then

1. if I is a left ideal of $R,$ then I is a quasi-ideal of $R,$
2. if I_i is a quasi-ideal of $R,$ then $\bigcap_{i\in\beta} I_i$ is a quasi-ideal of $R,$
3. if A is a left ideal and B a right ideal of $A \cap B,$ then $BA \subseteq \langle BA \rangle \subseteq A \cap B$ and $A \cap B$ is a quasi-ideal of $R,$
4. $\langle XR \rangle$ is a left ideal of $R,$

Proof. 1. Suppose that I is a left ideal of $R.$ Then by Theorem 3.5, we get $RI \subseteq \langle RI \rangle \subseteq I,$ so $RI \cap IR \subseteq \langle RI \rangle \cap \langle IR \rangle \subseteq \langle RI \rangle \subseteq I.$ Hence I is a quasi-ideal of $R.$

2. Suppose that I_i is a quasi-ideal of $R.$ Then $RI_i \cap I_i R \subseteq I_i.$ Since

$$\begin{aligned} R\left(\bigcap_{i\in\beta} I_i\right) \cap \left(\bigcap_{i\in\beta} I_i\right) R &= \left(\bigcap_{i\in\beta} RI_i\right) \cap \left(\bigcap_{i\in\beta} I_i R\right) \\ &\subseteq RI_i \cap I_i R \\ &\subseteq I_i \end{aligned}$$

we get $R\left(\bigcap_{i\in\beta} I_i\right) \cap \left(\bigcap_{i\in\beta} I_i\right) R \subseteq \bigcap_{i\in\beta} I_i.$ Hence $\bigcap_{i\in\beta} I_i$ is a quasi-ideal of $R.$

3. Suppose that A is a left ideal and B a right ideal of $R.$ Then $BA \subseteq RA \subseteq A$ and $BA \subseteq BR \subseteq B$ so that $BA \subseteq A \cap B.$ Let $x \in \langle BA \rangle.$ Then $x = \sum_{i=1}^{n} b_i a_i$ where $b_i a_i \in BA$ so that $b_i a_i \in A$ and $b_i a_i \in B.$ But A, B are LA-subsemigroups of $R,$ $x = \sum_{i=1}^{n} b_i a_i \in A$ and $x = \sum_{i=1}^{n} b_i a_i \in B$ so it implies that $x \in A \cap B.$ Thus $BA \subseteq \langle BA \rangle \subseteq A \cap B.$ To show that $A \cap B$ is a quasi-ideal of $R.$ Consider

$$\begin{aligned} \langle R(A \cap B) \rangle &= \langle RA \cap BR \rangle \\ &\subseteq \langle RA \rangle \\ &\subseteq A \end{aligned}$$

and

$$\begin{aligned} \langle (A \cap B) R \rangle &= \langle RA \cap BR \rangle \\ &\subseteq \langle BR \rangle \\ &\subseteq B. \end{aligned}$$

Then $\langle R(A \cap B) \rangle \cap \langle (A \cap B) R \rangle \subseteq A \cap B.$ Hence $A \cap B$ is a quasi-ideal of $R.$

4. To show that $R\langle XR \rangle \subseteq \langle R(XR) \rangle.$ Let $rx \in R\langle XR \rangle.$ Then

$$rx = r\sum_{i=1}^{n}(x_i r_i) = \sum_{i=1}^{n} r(x_i r_i) = \sum_{i=1}^{n} x_i(rr_i) \in \langle XR \rangle$$

so that $R\langle XR\rangle \subseteq \langle R(XR)\rangle$. Consider $R\langle XR\rangle \subseteq \langle R(XR)\rangle = \langle X(RR)\rangle = \langle XR\rangle$. Thus $\langle XR\rangle$ is a left ideal of R.

4. BI-IDEALS IN LEFT ALMOST SEMIRINGS

We start with the following theorem that gives a relation between quasi and bi-ideals in a left almost semiring. Our starting point is the following definition:

Definition 4.1. Let R be a left almost semiring. A left almost subsemiring B of R is called a bi-ideal of R if $(BR)B \subseteq B$.

Theorem 4.2. Let R be a left almost semiring with left identity and A a left ideal of R. If B is a quasi-ideal of A, then B is a bi-ideal of R.

Proof. Let B be a quasi-ideal of A. Consider

$$\begin{aligned}
(BA)B &= (B(AA))B \\
&= (A(BA))B \\
&= (B(BA))A \\
&= (B(BA))(AA) \\
&= (BA)((BA)A) \\
&= (A(BA))(AB) \\
&= (B(AA))(AB) \\
&= (BA)(AB) \\
&\subseteq B.
\end{aligned}$$

Then B is a bi-ideal of R.

Theorem 4.3. Let R be a left almost semiring with left identity. If B is a quasi-ideal of R, then B is a bi-ideal of R.

Proof. Let B be a quasi-ideal of R. Consider

$$\begin{aligned}
(BR)B &= (B(RR))B \\
&= (R(BR))B \\
&= (B(BR))R \\
&= (B(BR))(RR) \\
&= (BR)((BR)R) \\
&= (R(BR))(RB) \\
&= (B(RR))(RB) \\
&= (BR)(RB) \\
&\subseteq B.
\end{aligned}$$

Then B is a bi-ideal of R.

Proposition 4.4. Let R be a left almost semiring with a left identity. If B is a bi-ideal of $R,$ then $\langle B^2 \rangle$ is a bi-ideal of $R.$

Proof. Let B is a bi-ideal of $R.$ By Lemma 3.3, we have $\langle B^2 \rangle$ is a left almost subsemigroup of $(R,+).$ Then, by the Definition of a bi-ideal, we get

$$\begin{aligned}\langle B^2 \rangle \langle B^2 \rangle &\subseteq \langle B^2 B^2 \rangle \\ &= \langle (BB)(BB) \rangle \\ &\subseteq \langle BB \rangle \\ &= \langle B^2 \rangle\end{aligned}$$

and

$$\begin{aligned}(\langle B^2 \rangle R)\langle B^2 \rangle &= (\langle B^2 \rangle \langle R \rangle)\langle B^2 \rangle \\ &\subseteq \langle B^2 R \rangle \langle B^2 \rangle \\ &\subseteq \langle (B^2 R) B^2 \rangle \\ &\subseteq \langle ((BB)R)B \rangle \\ &\subseteq \langle ((BR)B)B \rangle \\ &= \langle BB \rangle \\ &= \langle B^2 \rangle .\end{aligned}$$

Thus $\langle B^2 \rangle$ and is a bi-ideal of $R.$

Theorem 4.5. Let R be a left almost semiring with a left identity. If A is a left ideal and B is a bi-ideal of $R,$ then $\langle BA \rangle$ and $\langle A^2 B \rangle$ are bi-ideals of $R.$

Proof. By Lemma 3.3, we have $\langle BA \rangle$ and $\langle A^2 B \rangle$ are left almost subsemigroup of $(R,+).$ Consider

$$\begin{aligned}\langle BA \rangle \langle BA \rangle &\subseteq \langle (BA)(BA) \rangle \\ &= \langle (BB)(AA) \rangle \\ &\subseteq \langle BA \rangle\end{aligned}$$

and

$$\begin{aligned}(\langle BA \rangle R)\langle BA \rangle &= (\langle BA \rangle \langle R \rangle)\langle BA \rangle \\ &\subseteq \langle (BA)R \rangle \langle BA \rangle \\ &\subseteq \langle ((BA)R)(BA) \rangle \\ &= \langle ((BA)B)(RA) \rangle \\ &\subseteq \langle ((BA)B)A \rangle \\ &\subseteq \langle BA \rangle .\end{aligned}$$

that is, $\langle BA \rangle$ is a bi-ideal of $R.$ To show that $\langle A^2 B \rangle$ is a bi-ideal of $R,$ let consider

$$\begin{aligned}\langle A^2 B \rangle \langle A^2 B \rangle &\subseteq \langle (A^2 B)(A^2 B) \rangle \\ &= \langle (A^2 A^2)(BB) \rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle A^2B^2 \rangle \\
&= \langle A^2(eB^2) \rangle \\
&\subseteq \langle A^2(RB^2) \rangle \\
&= \langle A^2(R^2B^2) \rangle \\
&= \langle A^2(B^2R^2) \rangle \\
&= \langle B^2(A^2R^2) \rangle \\
&= \langle (R^2A^2)B^2 \rangle \\
&= \langle (RA^2)B^2 \rangle \\
&= \langle (A(RA))B^2 \rangle \\
&\subseteq \langle (AA)B \rangle \\
&= \langle A^2B \rangle
\end{aligned}
$$

and

$$
\begin{aligned}
(\langle A^2B \rangle R)\langle A^2B \rangle &= (\langle A^2B \rangle \langle R \rangle)\langle A^2B \rangle \\
&\subseteq \langle (A^2B)R \rangle \langle A^2B \rangle \\
&\subseteq \langle ((A^2B)R)(A^2B) \rangle \\
&= \langle ((A^2B)(RR))(A^2B) \rangle \\
&= \langle ((A^2R)(BR))(A^2B) \rangle \\
&= \langle ((A^2R^2)(BR))(A^2B) \rangle \\
&= \langle ((R^2A^2)(BR))(A^2B) \rangle \\
&= \langle ((A(R^2A))(BR))(A^2B) \rangle \\
&\subseteq \langle ((AA)(BR))(A^2B) \rangle \\
&= \langle (A^2(BR))(A^2B) \rangle \\
&= \langle (A^2A^2)((BR)B) \rangle \\
&\subseteq \langle (AA)B \rangle \\
&= \langle A^2B \rangle.
\end{aligned}
$$

Hence $\langle A^2B \rangle$ is a bi-ideal of R.

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