Series Solutions to a Class of Initial Value Problems with Space-Fractional Derivatives

Songabsorn Manorod[1], Duangkamol Poltem[2] and Araya Wiwatwanich[3*]

[1] Department of Mathematics, Faculty of Science, Burapha University
Chonburi, Thailand

[2] Department of Mathematics, Faculty of Science, Burapha University
Chonburi, Thailand

[3] Department of Mathematics, Faculty of Science, Burapha University
Chonburi, Thailand

[*] *Corresponding author's email: arywwwn [AT] gmail.com*

ABSTRACT— In this paper, we propose a semi-analytic method to provide Taylor series solutions to the initial value problems of space-fractional differential equations in the form

$$u_t(x,t) = F[u] + g(x),$$

where $F[u] = F(u, \partial_x^{q_1}, \partial_x^{q_2}, ..., \partial_x^{q_n})$ ***is any linear operator of finite arguments and*** $q_1, q_2, ..., q_n \in \mathbb{R} - \{0\}$. ***Some examples have been demonstrated by using the technique of the proposed method.***

Keywords— Partial Caputo Derivative, Fractional Differential Equations

---

1. INTRODUCTION

Nowadays, the fractional differential equations (FDEs) have been played a very important role in various fields of Sciences and Engineering. Discrete fraction-related problems have also been studied recently in the context of graph theory [1]. Several numerical or analytical methods have been modified to solve fractional equations, for examples, differential transform method (DTM) and its generalizations [2], several integral transform methods [3], variational iteration method (VIM) [4], and also Adomian decomposition method (ADM) [5]. In 2011, Kutafina [6] proposed a distinct technique to obtain series solutions for initial value problems of the form

$$u_t(x,t) = F[u] + g(x), \tag{1}$$

where $F[u] = F(u, u_x, u_{xx}, ..., u_{x^n})$ is any linear operator of finite arguments. This paper aims to solve initial value problems of the form (1) with linear operator $F[u] = F(u, \partial_x^{q_1}, \partial_x^{q_2}, ..., \partial_x^{q_n})$ and $q_1, q_2, ..., q_n \in \mathbb{R} - \{0\}$ by using the technique in [6].

The rest of paper has been composed of the followings. In the next section, we introduce definition of the partial Caputo derivative and some notations used in this context. In Section 3, we propose the theorem providing series solution of the linear fractional equations. Some examples are showed in Section 4. The conclusion is discussed in the last section.

2. DEFINITION AND NOTATIONS

Fractional derivative is defined in many ways such as Grünwald-Letnikov derivative, Riemann-Liouville derivative, and Caputo derivative, see [7]. For initial value problems, Caputo derivative is the most widely used because it requires initial conditions of integer-order as normal. In this section, we give the definition of the partial Caputo derivative which is used the proposed method.

Definition 1 Let m be a non-negative integer and n be a positive integer. The partial Caputo derivative of order $q+m$ for a 2-variable function $u(x,t)$ is defined as

$$\partial_{x,t}^{q+m}u(x,t):=\frac{\partial^q}{\partial x^q}\left(\frac{\partial^m}{\partial t^m}u(x,t)\right) \tag{2}$$

$$=\frac{1}{\Gamma(n-q)}\int_0^x (x-\xi)^{n-q-1}\frac{\partial^n}{\partial \xi^n}\left(\frac{\partial^m}{\partial t^m}u(\xi,t)\right)d\xi, \tag{3}$$

where $n-1<q<n$.

Remark 1 If $q=n$, a non-negative integer, we regard $\partial_{x,t}^{q+m}u(x,t)$ as a normal integer order partial derivative $\frac{\partial^{q+m}}{\partial x^q t^m}u(x,t)$.

Remark 2 If $u(x,t)$ of Definition 1 is a real analytic function on its domain, we can interchange the order m,n of partial derivative for 2-variable function $u(x,t)$, i.e.,

$$\frac{\partial^n}{\partial \xi^n}\left(\frac{\partial^m}{\partial t^m}u(\xi,t)\right)=\frac{\partial^m}{\partial t^m}\left(\frac{\partial^n}{\partial \xi^n}u(\xi,t)\right). \tag{4}$$

In this case, we also have

$$\partial_{x,t}^{q+m}u(x,t)=\partial_{t,x}^{m+q}u(x,t). \tag{5}$$

For convenience, we introduce some essential notations which are used locally in this paper. Each notation is defined with some additional detail as in Table 1.

Table 1: Notations used in this paper

Symbol	Description	Note
$(\cdot)_x^q$ or $\partial_x^q(\cdot)$	$\partial_{x,t}^{q+0}(\cdot)$, for $q\in \mathbb{R}-\{0\}$	example: $u_x^{1/2}=\frac{\partial^{1/2}}{\partial x^{1/2}}u(x,t)$
$(\cdot)_t$ and $(\cdot)_{tt}$	$\partial_{x,t}^{0+1}(\cdot)$ and $\partial_{x,t}^{0+2}(\cdot)$	$(\cdot)_t\equiv\frac{\partial(\cdot)}{\partial t}$ and $(\cdot)_{tt}\equiv\frac{\partial^2(\cdot)}{\partial t^2}$
$F'[u]$	$\frac{\partial}{\partial t}F[u]$	$F'[u_0]=\left[\frac{\partial}{\partial t}F[u]\right]_{t=0}$
$F''[u]$	$\frac{\partial^2}{\partial t^2}F[u]$	$F''[u_0]=\left[\frac{\partial^2}{\partial t^2}F[u]\right]_{t=0}$
$F^{(k)}[u]$	$\frac{\partial^k}{\partial t^k}F[u]$, for $k\geq 3$	$F^k[u_0]=\left[\frac{\partial^k}{\partial t^k}F[u]\right]_{t=0}$

3. RESULTS

Let $F[u]=F(u,\partial_x^{q_1},\partial_x^{q_2},...,\partial_x^{q_n})$ is any linear operator of finite arguments and $q_1,q_2,...,q_n\in\mathbb{R}-\{0\}$. Consider the following one-dimensional equation in the form

$$u_t(x,t)=F[u]+g(x), \tag{6}$$

with the initial condition

$$u(x,0)=f_0(x). \tag{7}$$

The Taylor series solution to (6) can be derived from the following theorem.

Theorem 1 Let $F[u]$ be analytic of its arguments. If there exists a real analytic solution $u(x,t)$ of the initial value problem (1) and (2), it can be expressed in the form

$$u(x,t) = a_0 + a_1 t + a_2 \frac{t^2}{2} + a_3 \frac{t^3}{3!} + \cdots + a_n \frac{t^n}{n!} + \cdots, \tag{8}$$

where

$$\begin{aligned} a_0 &= f_0(x), & a_3 &= F''[u_0], \\ a_1 &= g(x) + F[u_0], & &\vdots \\ a_2 &= F'[u_0], & a_n &= F^{(n-1)}[u_0], \\ & & &\vdots \end{aligned} \tag{9}$$

Proof. Firstly, we can rewrite the form (6) with the operator $L(\cdot) = \frac{\partial(\cdot)}{\partial t}$

$$L(u(x,t)) = F[u] + g(x). \tag{10}$$

Then we take the inverse operator $L^{-1}(\cdot) = \int_0^t (\cdot)\, dt$ into (10) to have

$$u(x,t) = g(x)t + f_0(x) + \int_0^t F[u]dt. \tag{11}$$

Expanding the Taylor series for $F[u]$ about $t = 0$ leads us to

$$F[u] = F[u_0] + F'[u_0] + F''[u_0]\frac{t^2}{2!} + \cdots + F^{(n)}[u_0]\frac{t^n}{n!} + \cdots. \tag{12}$$

Substituting (12) into (11) yields

$$u(x,t) = f_0(x) + g(x)t + F[u_0]t + F'[u_0]\frac{t^2}{2!} + F''[u_0]\frac{t^3}{3!} + \cdots + F^{(n-1)}[u_0]\frac{t^n}{n!} + \cdots. \tag{13}$$

Therefore, we obtain the solution

$$u(x,t) = a_0 + a_1 t + a_2 \frac{t^2}{2!} + a_3 \frac{t^3}{3!} + \cdots + a_n \frac{t^n}{n!} + \cdots,$$

where $a_0, a_1, a_2, \ldots$ are coefficients as claimed in (9).

4. ILLUSTRATIVE EXAMPLES

To demonstrate the workable technique of the method, we give some examples.

Example 1 Consider the following fractional initial value problem

$$u_t = u_x^{1/2}, \tag{14}$$

$$u(x,0) = x. \tag{15}$$

By applying theorem 1, we obtain

$$\begin{aligned} a_0 &= x, \\ a_1 &= g(x) + F[u_0] \\ &= (a_0)_x^{1/2} \\ &= \frac{x^{1/2}}{\Gamma(3/2)}, \end{aligned}$$

$$
\begin{aligned}
a_2 &= F'\left[u_0\right] \\
&= \left[F_{u_x^{1/2}} \cdot \left(u_x^{1/2}\right)_t\right]_{t=0} \\
&= 1 \cdot \left(a_1\right)_x^{1/2} \\
&= 1, \\
a_3 &= F''[u_0] \\
&= \left[F_{u_x^{1/2}} \cdot \left(u_x^{1/2}\right)_{tt}\right]_{t=0} + \left(u_x^{1/2}\right)_t \left[F_{(u_x^{1/2})^2} \cdot \left(u_x^{1/2}\right)_t\right]_{t=0} \\
&= 1 \cdot \left(a_2\right)_x^{1/2} + \left(a_1\right)_x^{1/2}\left[0 \cdot \left(a_1\right)_x^{1/2}\right] \\
&= 0, \\
a_4 &= F'''\left[u_0\right] \\
&= 0, \\
&\vdots
\end{aligned}
$$

Therefore, the exact solution of this problem is

$$u(x,t) = x + \frac{x^{1/2}}{\Gamma(3/2)} t + \frac{t^2}{2!}. \tag{16}$$

Example 2 Consider nonhomogeneous fractional differential equation

$$u_t\left(x,t\right) = u_x^{1/2} + x, \tag{17}$$

with the initial condition

$$u\left(x,0\right) = 3x. \tag{18}$$

Then we obtain a solution for (17) in the form of

$$u(x,t) = a_0 + a_1 t + a_2 \frac{t^2}{2!} + a_3 \frac{t^3}{3!} + \cdots + a_n \frac{t^n}{n!} + \cdots,$$

where the Taylor coefficients is given by

$$
\begin{aligned}
a_0 &= 3x, \\
a_1 &= g(x) + F[u_0] \\
&= x + \left(a_0\right)_x^{1/2} \\
&= x + \frac{3x^{1/2}}{\Gamma(3/2)}, \\
a_2 &= F'\left[u_0\right] \\
&= \left(a_1\right)_x^{1/2} \\
&= \frac{x^{1/2}}{\Gamma(3/2)} + 3, \\
a_3 &= F''[u_0] \\
&= \left(a_2\right)_x^{1/2} \\
&= 1,
\end{aligned}
$$

$$a_4 = 0,$$
$$\vdots$$

Therefore, we obtain the exact solution

$$u(x,t) = 3x + \left[x + \frac{3x^{1/2}}{\Gamma(3/2)} \right] t + \left[\frac{x^{1/2}}{\Gamma(3/2)} + 3 \right] \frac{t^2}{2!} + \frac{t^3}{3!}. \tag{19}$$

5. CONCLUSION

In this paper, we presented an alternative tool for solving fractional differential equations. This method provides approximate solutions of linear equations with space-fractional derivative in the form of Taylor's series. In some cases, we probably obtain the exact solution like the two demonstrated examples. For further work, the proposed method can be applied to solve the initial value problems of nonlinear operator. This method can also be improved to solve the initial value problems of second order in time with space-fractional derivatives or even time-fractional derivatives.

6. ACKNOWLEDGEMENT

This work was supported by the Faculty of Science, Burapha University and Science Achievement Scholarship of Thailand (SAST). The authors would like to thank the reviewers for their useful comments and discussion.

7. REFERENCES

[1] Shang Y, "Vulnerability of networks: Fractional percolation on random graphs", Physical Review E, vol. 89, 2014.
[2] Arikoglu A, Ozkol I, "Solution of fractional differential equations by using differential transform method", Chaos, Soliton and Fractals, vol. 34, pp.1473-1481, 2007.
[3] Aghili A, Motahhari A, "Integral transform method for solving non homogenous time fractional partial differential equations", Differential Equations and Control Processes, no. 4, pp.194-204, 2011.
[4] Das S, "Analytic solution of a fractional diffusion equation by variational iteration method", Computers and Mathematics with Applications, vol. 57, pp.483-487, 2009.
[5] Cheng J. F, Chu Y. M, "Solution to the linear fractional differential equation using Adomian Decomposition Method", Mathematics Problems in Engineering, pp.1-14, 2011.
[6] Kutafina E, "Taylor series for Adomian decomposition method", vol. 88, no. 17, pp.3677-3684, 2011.
[7] Das S, Functional Fractional Calculus, 2nd ed., Springer-Verlag, Berlin, Heldelbery, 2011.