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On the Merits of Two Different Topologies on the Dual of a Hilbert space

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ABSTRACT – ***Two topologies of a Hilbert space l_2 is considered in this work. To wit normed vector space, which is its natural topology and weak topology. Among other properties we showed that the closed unit ball S is not compact when l_2^* is given its topology as a Banach space. On the other hand S is compact when l_2^* is given its weak topology.***

Keywords: Hilbert space, Normed topology, Closed set, Weak topology, Compactness.

1 Introduction

The most basic mathematical structure for functional analysis is the Banach space. The richer structure of Hilbert spaces can lead to greater depth and insight. Certainly, the most elementary structure either than a Banach space or Hilbert space, but which still yields useful result is that of topological vector space [1]. We discuss two different topologies on a dual of a Hilbert space, some general theory and operators defined on them. It is in this regard that we have come to realize the need to have this research work .

Let R be the field of all real numbers. Given a Banach space V over R, let V^* be the dual of R. V^* is the space of all continuous linear functionals on V [2]. There are several ways of defining a topology on V^*.

If V is a Banach space, the closed unit ball may be compact. In a finite dimensional Banach space the closed unit ball is compact (by the Heine Borel theorem). However in an infinite dimensional Banach space, when V^* is given the normed topology, the closed unit ball in V^* is not compact. On the other hand when V^* is given the weak topology the closed unit ball in V^* is compact.

The space l_2 of sequences x_n of real numbers for which will be used to illustrate this idea.

2 Definitions, Theorems and Proofs

Theorem 2.1. *Let X be a topological space. Then these two statements on a subset V of X are equivalent*

1. *V is an open set in X*
2. *for every $x \in V$ there exist an open set G_x in X such that*
   $x \in G_x \subset V$

Proof

Suppose (a) is true. If $x \in V$, let $G_x = V$ then
$x \in G_x \subset V$. Thus $(a) \implies (b)$
Next, suppose (a) is true. For each $x \in V$ choose an open set $G_x \in V$ such that

$x \in G_x \subset V$ then $V = \underset{x \in V}{U} G_x$
It follows that V is open and follows that V is open in X.

Definition 2.1. Let X be a topological space. A collection F of subsets of X is called a subbasis of the topology of X if for every open set G in X and every $g \in G$ there exist finitely many elements $D_1, ..., D_n \in F$ such that $g \in \cap_{j=1}^{n} D_j \subset G$

2.1 Construction of subbasis

Let X be a set. Suppose F is a non-empty collection of subset of X such that $X = \underset{D \in F}{U} D$. Let T be the set of all subsets G of X such that for every $a \in G$ there exit finitely many elements $D_1, ..., D_n \in F$ such that $a \in \cap_{j=1}^{n} D_j \subset G$. Then T is a topology in X. T is the topology generated by F on X or F is said to be a subbasis of the topology of X [3].

Definition 2.2. Normed Topology
Given a normed vector space $(V, \|, \|)$. Let a collection T of V be defined as follows:
$G \in T$ if and only if for every $a \in G$ there exist a positive real number δ such that
$\{x \in V : \|x - a\| < \delta\} \subset G$. Then T is a topology in V. T is called the topology induced on V by the norm. This is the same as the topology induced on V by the metric d. Where $d(x, y) = \|x - y\|$.

Definition 2.3. Bounded Linear Mapping
Let X and Y be normed linear spaces over the scalar field, $\mathbb{K}$, and let $T : X \longrightarrow Y$ be a linear map. Then T is said to be bounded if there exist some constant $k \geq 0$ such that for each $x \in X$, $\|T(x)\| \leq k\|x\|$ [2]

An important result in functional analysis is the following theorem.

Theorem 2.2. *Let $T : X \longrightarrow Y$ be a linear mapping where X, Y are normed vector spaces. Then these three statements are equivalent*

1. *T is continuous at 0*
2. *T is bounded*
3. *T is uniformly continuous*

Proof
Suppose (1) is true. Choose a positive real number δ such that for
$x \in X$ $\|x - 0\| < \delta \implies \|T(x) - T(0)\| < 1$.
That is $\|x\| < \delta \implies \|T(x)\| < 1$.
Now suppose $\left\| \dfrac{\frac{1}{2}\delta z}{\|z\|} \right\| = \dfrac{1}{2}\delta \dfrac{\|z\|}{\|z\|} = \dfrac{1}{2}\delta < \delta$ $z \in X$ and $z \neq 0$.
And so $\left\| T\left(\dfrac{\frac{1}{2}\delta z}{\|z\|} \right) \right\| < 1 \implies \dfrac{\frac{1}{2}\delta}{\|z\|}\|T(z)\| < 1 \implies \|T(z)\| < \dfrac{2}{\delta}\|z\|$.
Let $k = \dfrac{2}{\delta}$. Then for every $v \in X$, $\|T(v)\| \leq k\|v\|$. Therefore T is bounded. Thus (1) $\implies$ (2).
Suppose (2) is true. Given $\varepsilon > 0$, Let $\delta = \dfrac{\varepsilon}{k}$. Then for $z, v \in X$,
$\|z - v\| < \dfrac{\varepsilon}{k}$
$\implies \|T(z) - T(v)\| = k\|T(z - v)\| < \varepsilon$.
Therefore T is uniformly continuous. Thus (2) $\implies$ (3).
(3) $\implies$ (1) by definition.

Definition 2.4. Orthogonal and Orthonormal Set
A set in an product space A is called an orthogonal set if $\langle x, y \rangle = 0$ for each $x, y \in S, x \neq y$. The set S is called orthonormal if it an orthogonal set and $\|x\| = 1$ for each $x \in S$ [4]

Theorem 2.3. *Heine-Borel*
Let n be a positive integer. Give $\mathbb{R}^n$ its Euclidean norm $\|, \|$ defined by $\|x\| =$ where $x = (x_1, ..., x_n)$. Then a subset B of $\mathbb{R}^n$ is compact if and only if B is closed and bounded [5].

2.2 Bolzano-Weierstrass property of Compact set in a normed vector space

If A is a compact set in a normed vector space $(V, \|, \|)$ and $\{a_n\}$ is a sequence in A, then there exist $b \in A$ and the subsequence a_{n_k} of a_n such that $a_{n_k} \longrightarrow b$ as $k \longrightarrow \infty$ [4].

Definition 2.5. A sequence $\{x_n\}$ in a normed vector space is said to converge if there is a point $p \in V$ with the following property: For every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies that $\|X_n - p\| < \varepsilon$.

Definition 2.6. Let $(V, \|, \|)$ be a normed vector space. A sequence X_n in V is called a Cauchy Sequence in $(V, \|, \|)$ if to every positive real number ε, there correspond a positive integer P such that $m > P$ and $n > P$ implies $\|X_m - X_n\| < \varepsilon$.

Definition 2.7. There is the weak topology of V^* which is the topology of V^* generated by all set of the form $U(x, G) = \{f \in V^* : f(x) \in G\}$ for each $x \in V$, and each open set $G \subset R$. Let the set $U(x, G)$ be a subbasis for the topology in V^*. This topology is called the weak topology in V^* determined by V.

3 Main Results

We now give an example of a closed and bounded subset which is not compact in an infinite dimensional Hilbert space and also look for a theorem that shows that it is compact when given weak topology.

3.1 Example

Let $S = \{x \in l_2 : \|x\| \leq 1\}$
Then S is a closed and bounded subset in the Hilbert space l_2, but S is not compact.
Proof
In l_2 define a sequence e_n as follows: $e_n(k) = \begin{cases} 0 & \text{if } k \neq n \\ 1 & \text{if } k = n \end{cases}$.
Then $\{e_n : n = 1, 2, 3, ...\}$ is an orthonormal set in l_2. Assume that S is compact in the normed vector space $(V^*, \|, \|_*)$. Then the sequence e_n has a convergent subsequence e_{n_k}. Choose a positive integer q such that $\|e_{n_j} - e_{n_k}\| < 1$ whenever $j \geq q$ and $k \geq q$.
Then there is a contradiction $\sqrt{2} = \|e_{n_q} - e_{n_{q+1}}\| < 1$. The assumption is false. Hence S is not compact.

Finally we look for a topology on l_2 for which S is compact in the following theorem.

Theorem 3.1. *If V is a Banach space and the dual V^* is given its weak topology, then the closed unit ball $S \subset V^*$ is compact [6].*
Proof
For each $x \in V$. Let R_x be the set of real numbers. The space $\Pi[R_x : x \in V]$ evidently contains V^*, and the topology in V^* is the one induced by the topology in $\Pi[R_x : x \in V]$. Let K be the subset of $\Pi[R_x : x \in V]$ of points whose x coordinate has absolute value not greater than $\|x\|$, for every $x \in V$. By the theorem (that is $X_\alpha, \alpha \in A$, are compact spaces, then $X = \Pi[X_\alpha : \alpha \in A]$ is compact), K is compact. We show that the closed unit ball $S \subset V^*$ is a weakly closed subset of K
Let, the weak closure of S in K. Then $\psi \in K$. We show that ψ is linear. For this, let $x, y, x + y \in V$. Let $\varepsilon > 0$. By the definition of the topology $\Pi[R_x : x \in V]$, there is an $x^* \in V^*$ such that $|x^*(x) - \psi(x)| < \varepsilon$, $|x^*(y) - \psi(y)| < \varepsilon$ and $|x^*(x + y) - \psi(x + y)| < \varepsilon$. Since $x^*(x + y) = x^*(x) + x^*(y)$, it follows that $|(x + y) - \psi(x) - \psi(y)| < 3\varepsilon$. Since this holds for every $\varepsilon > 0$, $\psi(x + y) = \psi(x) + \psi(y)$. It is just as easy to show that for every $x \varepsilon V$ and every $a \in R$, $\psi(ax) = a\psi(x)$. Since $|(x)| \leq \|x\|$, for every $x \in V$, it follows that $\psi \in B$. Hence S is closed [6].

4 Conclusion

The conclusion is that when l_2^* is given its normed topology then l_2^* is isomorphic to l_2. In this case $\{x \in l_2^* : \|x\| \leq 1\}$ is not compact. On the other hand l_2^* can be given the weak topology and in this case $\{x \in l_2^* : \|x\| \leq 1\}$ is compact. It must be emphasized that the normed topology is the natural topology of V^*. There are times when the weak topology is needed. On the same vector space several unequal topologies may be defined and when you want to choose one you have to decide your purpose.

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