On Grid Method for Entropy Solution of the Problem of Simultaneous Motion of Two-Phase Fluid in a Natural Reservoir

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ABSTRACT— *In this paper a method for obtaining an exact and numerical solution of the initial and initial-boundary value problems for a first order partial differential equation with a non-convex state function is suggested, which models the macroscopic motion of the two- phase fluids in a porous medium. For this aim, an auxiliary problem is introduced such a way that it has some advantages over the main problem, and it is equivalent to the main problem in a definite sense. By use of this auxiliary problem it is proved that the exact and numerical solutions of the investigated problems satisfy the entropy condition in the sense of Oleinik. To make use of this auxiliary problem, a method for fixing the location of shock which appears in the solution of the main problem and its evolution in time is offered. The proposed auxiliary problem permits us also to prove convergence in the meaning of a numerical solution to the exact solution of the main problem. Besides, the auxiliary problem permits us to write the higher sensitive and the higher order numerical scheme with respect to time variable whose solution expresses all the physical properties of the problem accurately. Using the suggested algorithms, two laboratory experiments were carried out.*

Keywords— Buckley-Leverett's problem, Non-convex state function, Entropy solution, Numerical solution in a class of discontinuous functions

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1. INTRODUCTION

It is known that the mathematical model of the simultaneous motion of the petrol and water in a porous medium are based on the two fundamental laws of physics, one of the laws is conservation of petrol and water mass and the other one is the Darcy's law, [11]. Within the limits of some physical assumptions, the motion of two-phase fluid in the porous medium is described by the following system of equations

$$m\frac{\partial \sigma_\ell}{\partial t} + \mathrm{div}(u_\ell) = 0, \tag{1}$$

$$u_\ell = -\frac{kk_\ell(\sigma)}{\mu_\ell}\,\mathrm{grad}\,P_\ell, \tag{2}$$

$$P_p - P_w = \wp_k(\sigma_w), \tag{3}$$

$$\sigma_p + \sigma_w = 1, \quad (\ell = w, p). \tag{4}$$

Here, σ_ℓ and $P_\ell, (\ell = w, p)$ are unknown functions of saturations of the water and oil phases and pressure, respectively, u_ℓ are speed of motion of water and oil, $k_\ell, (\ell = w, p)$ are the relative permeabilities for oil and water, $\wp_k$ is the capillary pressure between water and oil, m and k are the porosity and permeability, respectively. For the

sake of simplicity, let $\sigma_w \equiv \sigma$. In accordance with the nature of exploitation, the initial and boundary conditions are added to the system of equations (1)-(4).

Taking into account (3) from (2) we get

$$k\left(\frac{k_w(\sigma)}{\mu_w} + \frac{k_p(\sigma)}{\mu_p}\right)\operatorname{grad} P_w + \frac{k_p(\sigma)}{\mu_p}\operatorname{grad} \wp_k(\sigma) = -u,$$

where $u = u_w + u_p$. From here

$$\operatorname{grad} P_w = \frac{-u}{k\left(\frac{k_w(\sigma)}{\mu_w} + \frac{k_p(\sigma)}{\mu_p}\right)} - \frac{\frac{k_p(\sigma)}{\mu_p}\operatorname{grad}\wp_k(\sigma)}{k\left(\frac{k_w(\sigma)}{\mu_w} + \frac{k_p(\sigma)}{\mu_p}\right)}. \tag{5}$$

Substituting the last expression into the equation of conservation law of water we obtain

$$m\frac{\partial\sigma}{\partial t} + \operatorname{div}\left\{-\frac{kk_w(\sigma)}{\mu_w}\left[\frac{-u}{k\left(\frac{k_w(\sigma)}{\mu_w} + \frac{k_p(\sigma)}{\mu_p}\right)} - \frac{\frac{k_p(\sigma)}{\mu_p}\operatorname{grad}\wp_k(\sigma)}{k\left(\frac{k_w(\sigma)}{\mu_w} + \frac{k_p(\sigma)}{\mu_p}\right)}\right]\right\} = 0. \tag{6}$$

Using the Buckley-Leverett functions defined by

$$F_w(\sigma) = \frac{\frac{k_w(\sigma)}{\mu_w}}{\frac{k_w(\sigma)}{\mu_w} + \frac{k_p(\sigma)}{\mu_p}}, \quad F_p(\sigma) = \frac{\frac{k_p(\sigma)}{\mu_p}}{\frac{k_w(\sigma)}{\mu_w} + \frac{k_p(\sigma)}{\mu_p}}, \quad F_w(\sigma) + F_p(\sigma) = 1$$

the equation (6) can be written in the form

$$m\frac{\partial\sigma}{\partial t} + \operatorname{div}\left(uF_w(\sigma)\right) + \operatorname{div}\left(\frac{k_p(\sigma)}{\mu_p}F_w(\sigma)\operatorname{grad}\wp_k(\sigma)\right) = 0, \tag{7}$$

[3], [11].

The problem (1)-(4) taking into account capillary pressure with corresponding initial and boundary conditions has been studied, [1], [18].

If the capillary pressure is $\wp_k(\sigma) = 0$, and in one-dimensional case the equation (7) turns into a nonlinear equation of the first order, [3]

$$m\frac{\partial\sigma}{\partial t} + u(t)\frac{\partial F_w(\sigma)}{\partial x} = 0. \tag{8}$$

The solution of the equation (8), obtained by using the method of characteristics, has an implicit form as

$$\sigma(x,t) = f\left(x - \frac{F_w'(\sigma)}{m}\int_0^t u(\eta)d\eta\right), \tag{9}$$

where f is any differentiable function, [4], [7], [9]. But, from (9), it is often impossible to obtain an explicit formula for the unknown function. We will call the obtained functional relation (9) as the alternative form of the equation (8). It is easily shown that by means of the transform $\tau = \dfrac{1}{m} \displaystyle\int_0^t u(\eta)d\eta$ the equation (8) can be rewritten in more simple form

$$\frac{\partial \sigma}{\partial t} + \frac{\partial F_w(\sigma)}{\partial x} = 0. \tag{10}$$

We assume that this transform has taken place before, and later on, the above equation (10) will be the subject of interest.

In the case where the initial function has both positive and negative slopes, or is piecewise constant, in general, if $\sigma(x,0) \in L_\infty(\Re^2)$ and $F_w''(\sigma) > 0$ (or $F_w''(\sigma) < 0$) then it is noted that [7], [8], [13], [22] the Cauchy problem have multi-valued solutions from which the physically efficient solution can be obtained by imposing the so-called entropy condition. In [13] under the assumption that $F_w''(\sigma)$ does not change sign, a method for obtaining the weak solution satisfying the entropy condition is proposed.

In this paper, we investigate the Cauchy and boundary value problems for a one-dimensional first-order nonlinear wave equation and offer a method for obtaining the unique exact and numerical solution in a class of discontinuous functions when $F_w''(\sigma)$ has alternative signs.

2. THE CAUCHY PROBLEM

Let $R^2(x,t)$ be the Euclidean space of points (x,t), and let $Q_T = \{x \in (-\infty, \infty), 0 \leq t \leq T\} \subseteq R^2(x,t)$.

In this section, we will construct the exact solution of the equation (10) with

$$\sigma(x,0) = \sigma_0(x) \tag{11}$$

initial condition and investigate some properties of this solution. Here, $\sigma_0(x)$ is a known continuous differentiable function with a compact support having both positive and negative slopes.

Suppose that the function $F_w(\sigma)$ is known and satisfies the conditions:

(i) $F_w(\sigma)$ is a twice continuously differentiable and bounded function for bounded σ;

(ii) $F_w'(\sigma) \geq 0$ for $\sigma \geq 0$;

(iii) $F_w''(\sigma)$ is a function with alternating signs i.e., F_w has convex and concave parts.

A solution of the problem (10), (11) can easily be constructed by the method of characteristics and it has the form

$$\sigma(x,t) = \sigma_0(\xi), \tag{12}$$

where

$$\xi = x - F_w'(\sigma)t \tag{13}$$

is the spatial coordinate moving with speed $F_{w'}(\sigma)$.

From (12) and (13) we have

$$\frac{\partial \sigma(x,t)}{\partial x} = \frac{\sigma_0'(\xi)}{(1 + \sigma_0'(\xi)F_w''(\sigma)t)}, \quad \frac{\partial \sigma(x,t)}{\partial t} = -\frac{\sigma_0'(\xi)F_{w'}'(\sigma)}{(1 + \sigma_0'(\xi)F_w''(\sigma)t)}.$$

The first formula explains the slope of the profile $\sigma(x,t)$ at the point (x,t) in terms of the slope of the initial profile at $(x = \xi, t = 0)$. If $\sigma_0' < 0$ and $F_w'' > 0$, (or $\sigma_0' > 0$ and $F_w'' < 0$) then for $t = \dfrac{-1}{\sigma_0'(\xi)F_w''(\sigma)}$ we have $\sigma_x(x,t) = \infty$. At these points $\sigma_t(x,t)$ also becomes infinite. Therefore, the problem (10), (11) does not have a classical solution.

Definition 1. The nonnegative function $\sigma(x,t)$ satisfying the condition (11) is called the generalized (weak) solution of the problem (10), (11) if the following integral relation

$$\iint_{Q_T}\{\varphi_t(x,t)\sigma(x,t)+\varphi_x(x,t)F_w(\sigma(x,t))\}dxdt+\int_{-\infty}^{\infty}\sigma(x,0)\varphi(x,0)dx=0 \tag{14}$$

holds for every test function $\varphi(x,t)$, which is defined and twice differentiable in the upper half plane and which is equal to zero for sufficiently large $t + |x|$.

2.1 The Auxiliary Problem

To determine the weak solution of the problem (10), (11), in accordance with [14], [15] and [19] the auxiliary problem

$$\frac{\partial v(x,t)}{\partial t}+F_w\left(\frac{\partial v(x,t)}{\partial x}\right)=0, \tag{15}$$

$$v(x,0)=v_0(x) \tag{16}$$

is introduced. Here, $v_0(x)$ is any absolutely continuous function satisfying the following equation

$$\frac{dv_0(x)}{dx}=\sigma_0(x). \tag{17}$$

The solution of the problem (15), (16) can easily be obtained, and has the form

$$v(x,t)=[\sigma F_{w'}(\sigma)-F_w(\sigma)]t+v_0(\xi), \quad \xi=x-F_w'(\sigma)t. \tag{18}$$

By calculation, it can be easily shown that $\sigma(x,t)=\dfrac{\partial v(x,t)}{\partial x}$. It is not difficult to see that a soft solution is a generalized solution (10), so the following theorem is valid.

Theorem 1. Let $v(x,t)$ be the solution of the problem (15), (16), then

1^0. the function $\sigma(x,t)$ defined by

$$\sigma(x,t)=\frac{\partial v(x,t)}{\partial x} \tag{19}$$

is the generalized solution of the problem (10), (11);

2^0. $v(x,t)$ is an absolutely continuous function.

The problem (15), (16) has the following advantages:

(i) The regularity of $v(x,t)$ is higher than that of $u(x,t)$;

(ii) $\sigma(x,t)$ can be determined without using the derivatives $\dfrac{\partial\sigma}{\partial x}$ and $\dfrac{\partial\sigma}{\partial t}$, which are not defined along the curve of discontinuities.

2.2 Front Tracking

To obtain the location of the points of discontinuity which arise in the solution of the main problem we will use the facts that $\int_{-\infty}^{\infty}\sigma(x,t)dx=const$, and that this integral exists not only for multi-valued and continuous functions but also for single-valued piecewise continuous functions. This fact is a result of the equation (10) expressing the conservation law of mass. Let $E_1(t)$ denote the following integral $E_1(t)=\int_R \sigma(x,t)dx$.

Definition 2. $E_1(0)=\int_R \sigma(x,0)dx$ is said to be the critical value of $v(x,t)$.

Now we direct our attention to the question of obtaining the locations of jump points of $\sigma(x,t)$ and their propagation over time. As it was noted above, the solution of the auxiliary problem is not unique. Some additional conditions are required for finding a unique physically meaningful solution.

Definition 3. For any t, the geometrical position of the points, where $v(x,t)$ takes a critical value is called the front curve.

Let $x_f = x_f(t)$ be the equation of the discontinuity curve of $v(x,t)$. Concerning Definition 3 and (19), we get

$$v(x_f(t),t) = \int_{-\infty}^{x_f} \sigma(x,t)dx = E_1(0).$$

From the last relation we have

$$\frac{dx_f(t)}{dt} = \frac{[F_w(\sigma)]}{[\sigma]}\bigg|_{x=x_f(t)} \tag{20}$$

Here $[f]$ shows the shock of the function f at a point $x = x_0$, i.e. $[f] = f(x_0 + 0) - f(x_0 - 0)$.

Definition 4. The function $v_{tr}(x,t)$ defined by

$$v_{tr}(x,t) = \begin{cases} v(x,t), & v < E_1(0), \\ \\ E_1(0), & v \geq E_1(0) \end{cases} \tag{21}$$

is called the truncated solution of the problem (15), (16).

From Theorem 1, for the weak solution of the main problem (10), (11), we have $\sigma_{tr}(x,t) = \dfrac{\partial v_{tr}(x,t)}{\partial x}$. This means that a point of discontinuity for $\sigma(x,t)$ is one to the right of which the solution of the problem (10), (11) is equal to zero.

From (20), we have

$$t = \int_0^{x_f(t)} \frac{dx}{F_{w'}(\sigma)}.$$

If $\displaystyle\int_0^{x_f(t)} \frac{dx}{F_{w'}(\sigma)} < \infty$, this means that the jump in $\sigma(x,t)$ occurs within finite time.

Now we consider the relation $\dfrac{v(x,t) - v(x-a,t)}{a}$ for any $a > 0$

$$\frac{v(x,t) - v(x-a,t)}{a} = \frac{1}{a}\int_0^t [F(u(x,\tau)) - F(u(x-a,\tau))]d\tau$$

$$\leq \frac{1}{a}\int_0^T [F(u(x,\tau)) - F(u(x-a,\tau))]d\tau \leq \frac{E_2}{t}, \tag{22}$$

here $E_2 = \dfrac{2}{a}\sup_u F(u)$. This is the entropy condition in the sense of Oleinik [13], which shows the rate of spreading of characteristics. Hence $v(x,t)$ is the entropy solution of the problem (15), (16).

3. THE INITIAL-BOUNDARY VALUE PROBLEM

Until now, we found the weak solution of the Cauchy problem for a first-order nonlinear equation of the hyperbolic type. But, many important practical problems such as the displacement of oil by water in a porous medium are expressed by the initial-boundary value problem for the equation (10), [2], [3], [10].

The typical initial-boundary value problem describing the distribution of some signal in $D = \{x > 0, \ t > 0\}$ is

$$\frac{\partial \sigma}{\partial t} + \frac{\partial F_w(\sigma)}{\partial x} = 0, \tag{23}$$

$$\sigma(x,0) = \sigma_0(x), \tag{24}$$

$$\sigma(0,t) = \sigma_1(t). \tag{25}$$

Here, $\sigma_0(x)$ and $\sigma_1(t)$ are given functions, and $\sigma_1(0) > \sigma_0(0)$.

It is obvious that the solution of the problem (23)-(25) may be connected to the solutions of two Cauchy problems, for $F_w(\sigma)$ function which satisfies the conditions mentioned in section 2, when $F'_w(\sigma) > 0$. We introduce the following Cauchy problems:

$$\frac{\partial \sigma}{\partial t} + \frac{\partial F_w(\sigma)}{\partial x} = 0, \tag{26}$$

$$\sigma(x,0) = \sigma_0(x); \tag{27}$$

and

$$\frac{\partial \sigma}{\partial t} + \frac{\partial F_w(\sigma)}{\partial x} = 0, \tag{28}$$

$$\sigma(0,t) = \sigma_1(t). \tag{29}$$

The exact solution of the main problem (23)-(25) was constructed in [21], and has the form

$$\sigma(x,t) = \begin{cases} \sigma_0(\xi), & \dfrac{x}{t} > F'_w(\sigma_0), \\[2mm] G(\dfrac{x}{t}), & F'_w(\sigma_0) < \dfrac{x}{t} < F'_w(\sigma_1), \\[2mm] \sigma_1(\tau), & \dfrac{x}{t} < F'_w(\sigma_1). \end{cases} \tag{30}$$

Here $G(\xi)$ is the inverse function $F'_w(\sigma)$ over $[\sigma_0, \sigma_1]$. As it is known that, the solution (30) is a multi-valued function for any $x > 0$ and $t > 0$.

The weak solution of the problem (23)-(25) is defined as:

Definition 5. The nonnegative function $\sigma(x,t)$ satisfying the conditions (24), (25) is said to be a generalized solution of the problem (23)-(25) if the integral relation

$$\iint_D \{f_t(x,t)\sigma + f_x(x,t)F_w(\sigma)\}dxdt + \int_0^\infty \sigma(x,0)f(x,0)dx + \int_0^T F_w(\sigma(0,t))f(0,t)dt = 0 \tag{31}$$

is valid for any test functions $f(x,t)$ which vanishes for large x and $f(x,T) = 0$.

To obtain the generalized solution of the problem (23)-(25) in the sense of (31), according to [14] and [15], the following auxiliary problem, known as the second kind auxiliary problem

$$\frac{\partial v(x,t)}{\partial t} + F_w\left(\frac{\partial v(x,t)}{\partial x}\right) = 0, \tag{32}$$

$$v(x,0) = v_0(x), \tag{33}$$

$$\frac{\partial v(0,t)}{\partial x} = \sigma_1(t) \tag{34}$$

is introduced.

As stated above, we will split the problem (32)-(34) into following two Cauchy problems

$$\frac{\partial v(x,t)}{\partial t} + F_w\left(\frac{\partial v(x,t)}{\partial x}\right) = 0,$$

$$v(x,0) = v_0(x);$$

and

$$\frac{\partial v(x,t)}{\partial t} + F_w\left(\frac{\partial v(x,t)}{\partial x}\right) = 0,$$

$$\frac{\partial v(0,t)}{\partial x} = \sigma_1(t).$$

The solution of the problem (32)-(34) is constructed using by solutions of these problems

$$v(x,t) = \begin{cases} v_0(\xi) + \left(\sigma_0(\xi)F_w'(\sigma_0(\xi)) - F_w(\sigma_0(\xi))\right), & \frac{x}{t} > F_w'(\sigma_0), \\[2ex] \int G(\frac{x}{t})dx, & F_w'(\sigma_0) < \frac{x}{t} < F_w'(\sigma_1), \\[2ex] \left(-\frac{F_w(\sigma_1)}{F_w'(\sigma_1)x} + \sigma_1(\tau)\right)\sigma_1(\tau)x - \int_0^\tau F_w(\sigma_1(\tau_1)d\tau_1), & \frac{x}{t} < F_w'(\sigma_1). \end{cases} \tag{35}$$

3.1 Build-up of Shock

To build the shock which appears in the solution, we will use the second kind auxiliary equation

$$\frac{\partial}{\partial t}\int_0^x \sigma(\xi,t)d\xi = F_w(\sigma_1(t)) - F_w(\sigma(x,t)). \tag{36}$$

instead of (15).

As is obvious from equation (36), the function $\sigma(x,t)$ in this case may be discontinuous, too. On the other hand, we have

$$v(x,t) = \int_0^x \sigma(\eta,t)d\eta + \sigma(0,t) = \sigma_1(t) + \int_0^x \sigma(\eta,t)d\eta. \tag{37}$$

Due to the fact that the fluid is incompressible, the volume of pumped water to porous medium is equal to

$$v_{\text{medium}}(x) = \sigma_1(0) + \int_0^x \sigma_0(\eta)d\eta.$$

Let the number $v_{\text{medium}}(x)$ be a critical value of the function $v(x,t)$. We will also call the point of front $x_f(t)$ (or shock) the point when the function $v(x,t)$ takes the critical value. Hence, this follows

$$v(x_f(t),t) = \sigma_1(t) + \int_0^{x_f(t)} \sigma(\eta,t)d\eta = v_{medium}(x). \tag{38}$$

Taking into consideration Definition 4 and Theorem 1 we have

$$v_{ext}(x,t) = \begin{cases} v(x,t), & v(x,t) < v_{medium}(x), \\ \\ v_{medium}(x), & v(x,t) > v_{medium}(x), \end{cases} \tag{39}$$

and

$$\sigma_{tr}(x,t) = \frac{\partial v_{tr}(x,t)}{\partial x}. \tag{40}$$

As it is seen from (40), the front point is a point where the residual water saturation is equal σ_0 anywhere on its right.

From (38) for the $x_f(t)$ we have

$$\frac{dx_f(t)}{dt} = \left(\frac{\sigma_1(t)}{\sigma} - \frac{F_w(\sigma)}{\sigma} \right)\Bigg|_{x=x_f(t)}.$$

Taking into consideration (36) and (19), for value of $t > 0$ and $a > 0$ we can write

$$\frac{v(x,t) - v(x-a,t)}{a} = \frac{1}{a}\int_0^t [F_w(u(x-a,\tau)) - F_w(u(x,\tau))]d\tau$$

$$\leq \frac{1}{a}\int_0^T [F_w(u(x-a,\tau)) - F_w(u(x,\tau))]d\tau \leq \frac{E_3}{t},$$

here $E_3 = 2sup_u \dfrac{F(u)}{a}$. Hence, the solution of the problem (23)-(25) is the entropy solution.

4. GRID METHOD IN A CLASS OF DISCONTINUOUS FUNCTIONS

As it is known, the solution of the equation (10) has points of discontinuities whose the locations are unknown beforehand. The presence of these points does not permit us to approximate the equation (10) by the finite differences method. The algorithms writing without considering of such points may be lead to false results.

Various finite difference methods have been applied to find the solution of the Cauchy problem for equation (10), [6], [9], [10], [12], [16], [17], [21]. There are many finite differences schemes without considering of jump points arising in the solution that the required solutions are constructed by using same algorithms, [20]. Besides, using the characteristic method some hybrid numerical algorithms are developed [5], [6].

As it is known, when the equation (10) is approximated by the classical finite difference schemes right side of the equation added some numerical viscosity and as a result of this, computational propagation moves ahead of physical propagation.

In this section, using the above introduced auxiliary problem we will develop a numerical method to solve the problem (10), (11), and investigate some properties of the solution.

4.1 The Grid Method for the Cauchy Problem

To build the finite differences scheme, at first, the domain of definition of the problem is covered by the uniform grid with steps h and τ. The problem (15), (16) is approximated by the finite differences scheme at any point (i,k) as follows

$$V_{i,k+1} = V_{i,k} - \tau F_w\left(\frac{V_{i,k} - V_{i-1,k}}{h} \right), \tag{41}$$

$$V_{i,0} = v_0(x_i). \tag{42}$$

Here, $v_0(x_i)$ is any continuous solution of the differences equation $(V_0)_{\bar{x}} = \sigma_0(x_i)$. By simple calculation, we can show that

$$\Sigma_{i,k+1} = \frac{V_{i,k+1} - V_{i-1,k+1}}{h}. \tag{43}$$

$\Sigma_{i,k}$ and $V_{i,k}$ symbolize approximate values of $\sigma(x,t)$ and $v(x,t)$ at point (i,k) respectively.

With regard to (43), we can show that $\Sigma_{i,k}$ are solutions of algebraic equations

$$\Sigma_{i,k+1} = \Sigma_{i,k} - \frac{\tau}{h}\left(F_w\left(\Sigma_{i,k}\right) - F_w\left(\Sigma_{i-1,k}\right)\right). \tag{44}$$

Theorem 2. The quantity $E_1(t_k) = h\sum_i\left(\Sigma_{i,k}\right)$ is not dependent on time.

Proof. Multiplying both sides of (44) by h and summing with respect to i we have

$$h\sum_{i=-\infty}^{\infty}\Sigma_{i,k+1} = h\sum_{i=-\infty}^{\infty}\Sigma_{i,k}.$$

Definition 6. The $E_1(0) = h\sum_i\left(\Sigma_{i,0}\right)$ is said to be the critical value of $V_{i,k}$.

Definition 7. The grid function

$$V_{i,k}^{tr} = \begin{cases} V_{i,k}, & V_{i,k} < E_1(0) \\ \\ E_1(0), & V_{i,k} \geq E_1(0) \end{cases}$$

is said to be the truncated numerical solution of the problem (41), (42).

From Theorem 1, we have $\Sigma_{i,k}^{tr} = (V_{i,k}^{tr})_{\bar{x}}$, and this expression is called the truncated numerical solution of the considered problem. The suggested algorithm (41), (42) is very effective and economical from a computational point of view.

The finite differences analogy of the (36) is

$$h\sum_{j=1}^{i}U_{i,k} = \tau\sum_{v=1}^{k}\left[F\left(U_{q,v}\right) - F\left(U_{i,v}\right)\right], \tag{45}$$

here q is such number for that the $r = (i-q)h$ is valid. Let p be any positive integer, and consider

$$\frac{V_{i,k} - V_{i-p,k}}{p} = \frac{1}{p}\left\{\tau\sum_{v=1}^{k}\left[F\left(U_{q,v}\right) - F\left(U_{i,v}\right)\right] - \tau\sum_{v=1}^{k}\left[F\left(U_{q,v}\right) - F\left(U_{i-p,v}\right)\right]\right\}$$

$$= \frac{1}{p}\left\{\tau\sum_{v=1}^{k}\left[F\left(U_{i-p,v}\right) - F\left(U_{i,v}\right)\right]\right\} \leq \frac{1}{p}\int_0^T\left[F\left(u(x-a,t)\right) - F\left(u(x,t)\right)\right]dt \leq \frac{E_4}{t_k}, \tag{46}$$

where $a = (i-p)h$ and $E_4 = \frac{2}{p}\max_u F(u)$. Therefore, the numerical solution of the problem (41), (42) satisfies the entropy condition too. Considering (19), we rewrite (15) as

$$\frac{\partial v(x,t)}{\partial t} + F_w(\sigma(x,t)) = 0. \tag{47}$$

Using Runge-Kutta methods to the equation (47), we purpose more sensitive algorithms for the main problem with respect to τ.

4.2 The Grid Method for the Boundary Initial Value Problem

The finite differences analogy for (23)-(25) is

$$V_{i,k+1} = V_{i,k} - \tau F\left(\frac{V_{i,k} - V_{i-1,k}}{h}\right), \tag{48}$$

$$V_{i,0} = v_0(x_i), \tag{49}$$

$$\frac{V_{i,k} - V_{0,k}}{h} = u_1(t_k). \tag{50}$$

It is easily shown that the equality (43) is fulfilled for the problem (48)-(50). As above the truncated solution of this problem is written in the form

$$v_{ext}(x,t) = \begin{cases} V_{i,k}, & V_{i,k} \le \tilde{E}_3(t_k), \\ \\ \tilde{E}_3(t_k), & V_{i,k} > \tilde{E}_3(t_k), \end{cases} \tag{51}$$

here $\tilde{E}_3(t_k) = \tau \sum_{v=1}^{k} \left[F(u_1(0)) - F(u(x,0)) \right]$. Using the Theorem 1, we can find the truncated numerical solution of the main problem (48)-(50).

We can write analogies estimation of (45) kind for the solution of the problem (32)-(34), i.e., for the solution in question entropy condition is satisfied.

To find the numerical solution of the problem (23)-(25) as a matter of fact, we will use the second kind auxiliary problem which is equivalent to (23)-(25). But then, auxiliary problem in question (23)-(25) is a convenient tool as theoretical investigations by proof of convergence of the numerical solution to the exact solution of the main problem, and by study of some theoretical property of the solution as well.

For practical calculations of the numerical solution of (23)-(25), we will use the equation (36). Firstly, we approximate the integral included in (36) by

$$\int_0^x \sigma(\xi,t)d\xi = h\sum_{j=1}^{i} \Sigma_{j,k}. \tag{52}$$

By taking into account, (52) for the equation (36) we will write two kinds of difference schemes:

1) Explicit scheme

$$\Sigma_{i,k+1} = \Sigma_{i,k} + \frac{\tau}{h}\left[F_w(\sigma_1(t_k)) - F_w(\Sigma_{i,k})\right] - \sum_{j=1}^{i-1}(\Sigma_{j,k+1} - \Sigma_{j,k}). \tag{53}$$

This differences scheme is simple and to obtain the solution $\Sigma_{j,k+1}$ from (53) does not present any difficulty. But this scheme requires the severe constraints on the steps of grid.

To flee from this limitation, we will write

2) implicit scheme

$$\Sigma_{i,k+1} = \Sigma_{i,k} + \frac{\tau}{h}\left[F_w(\sigma_1(t_k+1)) - F_w(\Sigma_{i,k+1})\right] - \sum_{j=1}^{i-1}(\Sigma_{j,k+1} - \Sigma_{j,k}). \tag{54}$$

The differences scheme (54) is nonlinear with respect to $\Sigma_{j,k+1}$. For finding this solution we can apply following scheme:

a) Simple iteration

$$\Sigma_{i,k+1}^{(s+1)} = \Sigma_{i,k} + \frac{\tau}{h}\Big[F_w(\sigma_1(t_k+1)) - F_w(\Sigma_{i,k+1}^{(s)})\Big] - \sum_{j=1}^{i-1}(\Sigma_{j,k+1} - \Sigma_{j,k}). \tag{55}$$

b) Newton iteration

$$\Sigma_{i,k+1}^{(s+1)} = \Sigma_{i,k} + \frac{\tau}{h}\Big[F_w(\sigma_1(t_k)) - F_w(\Sigma_{i,k}^{(s+1)})\Big] - \sum_{j=1}^{i-1}(\Sigma_{j,k+1} - \Sigma_{j,k}). \tag{56}$$

To obtain the solution we represent it in form

$$\Sigma_{i,k+1}^{(s+1)} = \Sigma_{i,k+1}^{(s)} + \delta\Sigma_{i,k+1}^{(s)}.$$

Substituting the last relation in (56) and linearizing it, we have

$$\delta\Sigma_{i,k+1}^{(s)} = -\sum_{j=1}^{i-1}\delta\Sigma_{j,k+1}^{(s)} - \frac{\tau}{h}F_w'(\Sigma_{i,k+1}^{(s)})\delta\Sigma_{i,k+1}^{(s)} + \Sigma_{i,k+1} - \Sigma_{i,k+1}^{(s)})$$

$$-\frac{\tau}{h}F_w(\Sigma_{i,k+1}^{(s)}) - \sum_{j=1}^{i-1}(\Sigma_{j,k+1} - \Sigma_{j,k}) - \frac{\tau}{h}F_w(\sigma(0, t_{k+1})). \tag{57}$$

It is obvious from (57) that this algorithm is economical and efficient from a computational point of view and it allows us to find the solution $\delta\Sigma_{i,k+1}^{(s)}$ easily.

4.3 Consistency and Convergence

Now we will show that the difference scheme (44) is monotone. For this, (44) let us rewrite in form

$$U_{i,k+1} = H\big(U_{i-1,k}, U_{i,k}\big),$$

where $H\big(U_{i-1,k}, U_{i,k}\big) = U_{i,k} + \frac{\tau}{h}\big(F_w(U_{i-1,k}) - F_w(U_{i,k})\big).$ It is obvious that if

$$0 \le \frac{\tau}{h}F_w'(U_{i,k}) \le 1 \quad \text{then} \quad \frac{\partial H}{\partial U_{j,k}} \ge 0$$

for $j = i-1, i$. Hence, if the CFL condition is fulfilled, then the difference scheme (44) is monotone. The definition of a monotone scheme is actually equivalent to the following property;

$$\text{if } W_{i,k} \ge U_{i,k} \text{ for any } i \text{ then } W_{i,k+1} \ge U_{i,k+1}. \tag{58}$$

Theorem 3. Let $\{U_{i,k}\}$ be given set, if $\{U_{i,k+1}\}$ is solution set founded with a monotone scheme (44) then

$$\max_i\{U_{i,k+1}\} \le \max_i\{U_{i,k}\}; \left(\text{or } \min_i\{U_{i,k+1}\} \ge \min_i\{U_{i,k}\}\right). \tag{59}$$

Proof. Let $W_{i,k} = \max_i U_{i,k}$ for any i. From (44) we have $W_{i,k+1} = W_{i,k}$. As $W_{i,k} \ge U_{i,k}$, application of (58) gives $W_{i,k+1} = W_{i,k} \ge U_{i,k+1}$ and therefore $\max_i\{U_{i,k+1}\} \le \max_i\{U_{i,k}\}$. The second inequality in (59) follows similarly.

From (59) also follows

$$\max_i\{U_{i,k}\} \le \max_i\{U_{i,k-1}\} \le ... \le \max_i\{U_{i,0}\},$$

$$\min_i\{U_{i,k}\} \ge \min_i\{U_{i,k-1}\} \ge ... \ge \min_i\{U_{i,0}\}.$$

It is easily shown that the differences schema (48) is monotone, too. Indeed, under the CFL condition $\frac{\partial H_1}{\partial V_{j,k}} \geq 0$ for any $j, (j = i-1, i)$ here,

$$H_1\left(V_{i-1,k}, V_{i,k}\right) = V_{i,k} - \frac{\tau}{h} F_w\left(\frac{V_{i,k} - V_{i-1,k}}{h}\right).$$

Let $\varepsilon_{i,k}$ and $\delta_{i,k}$ be the errors of the approximations by the differences of the derivatives $\frac{\partial v(x,t)}{\partial x}$ and $\frac{\partial v(x,t)}{\partial t}$. Then (32) can be written as

$$v_t + \delta_{i,k} + F_w\left(v_{\bar{x}} + \varepsilon_{i,k}\right) = 0$$

or

$$v_t + F\left(v_{\bar{x}}\right) = \eta_{i,k}, \tag{60}$$

where $\eta_{i,k} = \delta_{i,k} + F'(v_{\bar{x}})\varepsilon_{i,k}$.

Now we will show that the difference scheme (60) is consistent. It is known that the suitable characteristic of continuity of the function $f(x)$ on any $[a,b]$ is its module continuity

$$\omega(\delta, f) \equiv \pi(f) = \sup_{|t-x|<\delta} | f(t) - f(x) |.$$

At first, we will show that $\varepsilon_{i,k} \to 0$ and $\delta_{i,k} \to 0$ if the steps of grid approach to zero. Indeed, since $u(x,t)$ is continuous

$$\varepsilon_{i,k} = \frac{\partial v(x_i, t_k)}{\partial x} - v_{\bar{x}} = \frac{\partial v(x_i, t_k)}{\partial x} - \frac{\partial v(x_i^*, t_k)}{\partial x} = u(x_i, t_k) - u(x_i^*, t_k) = \pi(u) \to 0, \quad x_i^* \in (x_i, x_{i+1})$$

and

$$\delta_{i,k} = \frac{\partial v(x_i, t_k)}{\partial t} - v_t = \frac{\partial v(x_i, t_k)}{\partial t} - \frac{\partial v(x_i, t_k^*)}{\partial t} = F_w(u(x_i, t_k)) - F_w(u(x_i, t_k^*))$$

$$= F_w'(\tilde{u})(u(x_i, t_k) - u(x_i, t_k^*)) = F_w'(\tilde{u})\pi(u) \to 0,$$

$t_i^* \in (t_k, t_{k+1}), \quad \tilde{u} \in (u(x_i, t_k), u(x_i, t_k^*))$, hence, $\eta_{i,k} \to 0$.

Subtracting (60) from this equality and writing $w_{i,k}$ for $v_{i,k} - V_{i,k}$, we have the following relation for $w_{i,k}$

$$w_t + F_w'(\tilde{u})w_{\bar{x}} = \eta_{i,k}, \tag{61}$$

$$w_{i,0} = \int_0^x u_0(\xi)d\xi - h\sum_{j=1}^{i-1} u_0(x_j) = w_i^{(0)} = O(h) \to 0, \quad (i = 1, 2, \ldots),$$

$$w_{1,k} - w_{0,k} = h\varepsilon_{0,k} = O(h) \to 0.$$

According to Theorem 3,

$$w_{i,k} = v_{i,k} - V_{i,k} \to 0,$$

that is, the numerical solution $V_{i,k}$ of problem (48)-(50) pointwise approaches to the solution of the problem (32)-(34).

Now, let us multiply (61) to $w_{\bar{x}}$ and sum with respect to i, k over grid $\omega_{\tau,h}$,

$$(w_{\bar{x}}, w_t)_{L_2(\omega_{\tau,h})} + (w_{\bar{x}}, F'_w(\tilde{u})w_{\bar{x}})_{L_2(\omega_{\tau,h})} = (w_{\bar{x}}, \eta_{i,k})_{L_2(\omega_{\tau,h})}$$

or

$$(R_{i,k}, F'_w(\tilde{u})R_{i,k})_{L_2(\omega_{\tau,h})} = (w_{i,k}, R_t)_{L_2(\omega_{\tau,h})} + (R_{i,k}, \eta_{i,k})_{L_2(\omega_{\tau,h})} \leq$$

$$\left\|w_{i,k}\right\|_{L_2(\omega_{\tau,h})}\left\|R_t\right\|_{L_2(\omega_{\tau,h})} + \left\|w_{i,k}\right\|_{L_2(\omega_{\tau,h})}\left\|\eta_{i,k}\right\|_{L_2(\omega_{\tau,h})}.$$

Here, the notations $R_{i,k} = u_{i,k} - U_{i,k}$ and $(f, g)_{L_2(\omega_{\tau,h})}$ is differences analogy of the inner production of the functions f and g, that is

$$(f, g)_{L_2(\omega_{\tau,h})} = \int_{D_T} f(x)g(x)dx.$$

From last inequality it is seen that $u_{i,k}$ converges to $U_{i,k}$ with the weighted $F'_w(\tilde{u})$ in the sense of $L_2(\omega_{\tau,h})$.

5. MODELING THE LABORATORY EXPERIMENTS

In order to model the laboratory experiments imitating the displacement of oil by water, it is necessary to find the functions of oil-water relative permeabilities. The functions in question are found by experimentation in general. The values of those parameters are essential to perform field-scale analysis. But, it is interesting how much the obtained functions accurately describe the physical properties of the studied problem.

It is known that, even if, we isolate a solid rock from the petroleum reservoir in order to carry out an experiment, changes occur in its own physical and chemical properties. In this situation the results of the mathematical model may not be equal to the physical phenomenon. On the other hand, during laboratory experiments only the wateriness period is observed. But, by obtaining the functions of oil-water relative permeabilities, it is necessary to take into account the waterless period, too.

In general, finding relative permeabilities of oil-water considering both waterless and wateriness periods of the displacement of oil by water is not a simple problem. In order to accurately model an experiment, it is necessary to find the relative permeabilities of oil-water in the whole period of the displacement.

In this section using the natural conditions which should satisfy the functions $k_w(\sigma)$, $k_p(\sigma)$, $F_w(\sigma)$ and $F_p(\sigma)$ a theoretical method to obtain the $k_w(\sigma)$, $k_p(\sigma)$ is proposed.

The conditions in question are

$$k_p(\sigma_1) = 1, \, k_p(\sigma_0) = 0, \, F'_w(\sigma_1) = 0, \tag{62}$$

$$k_w(\sigma_1) = 1, \, k_w(\sigma_0) = 0, \, F'_w(\sigma_0) = 0. \tag{63}$$

We will seek the functions $k_p(\sigma)$ and $k_w(\sigma)$ in the form

$$k_w(\sigma) = a_0 + a_1\sigma + a_2\,\sigma^2, \quad k_p(\sigma) = b_0 + b_1\sigma + b_2\,\sigma^2. \tag{64}$$

Here the a_i and b_i, $(i = 0,1,2)$ as yet are unknown constants.

Taking into account the conditions (62), (63) for obtaining unknown constants we get

$$a_0 + a_1\sigma_1 + a_2\,\sigma_1^2 = 0,$$

$$a_0 + a_1\sigma_0 + a_2\,\sigma_0^2 = 1,$$

$$a_1 + 2a_2\sigma_1 = 0;$$

and

$$b_0 + b_1\sigma_1 + b_2\,\sigma_1^2 = 1,$$

$$b_0 + b_1\sigma_0 + b_2\,\sigma_0^2 = 0,$$

$$b_1 + 2b_2\sigma_0 = 0.$$

Solving those systems of algebraic equations and substituting these obtained values in (64) we have

$$k_w(\sigma) = \frac{(\sigma - \sigma_1)^2}{\sigma_1^2 - \sigma_0^2}, \quad k_p(\sigma) = \frac{(\sigma - \sigma_0)^2}{\sigma_1^2 - \sigma_0^2}. \qquad (65)$$

Now, we will simulate the experiments which were done in the Department of Physico-Chemistry of Porous Medium of the Institute for the Study of Problem of Deep Oil and Gas Deposits of the Azerbaijan Academy of Sciences.

In the first experiment, a cylindrical pipe filled with unfiltered quartz sand is used as the porous medium. The length and cross section of the pipe are given as $l = 1.2m$, $S = 9.6 \cdot 10^{-4} m^2$, respectively. The permeability coefficient, k, is $2.22\mu m^2$, and the porosity (m) is 0.298. The transformer oil is used as the fluid model of which the viscosity is $47.9\,sP$, and the surface tension between the water and the fluid is 37 $\mu N/m$. The pipe from one end is attached to a water source whose gradient pressure (Δp) is $0.03mPa$. The residual water saturation is (s_0) 0.23. The duration of the experiment is 27 hours. The time of waterless period of the experiment was truncated about 14 hours and the whole time is equal to 14 hours. In waterless period degree of efficiency was computed as $\eta_{waterless} = 0.32$, but total degree of efficiency is equal to $\eta_{total} = 0.51$.
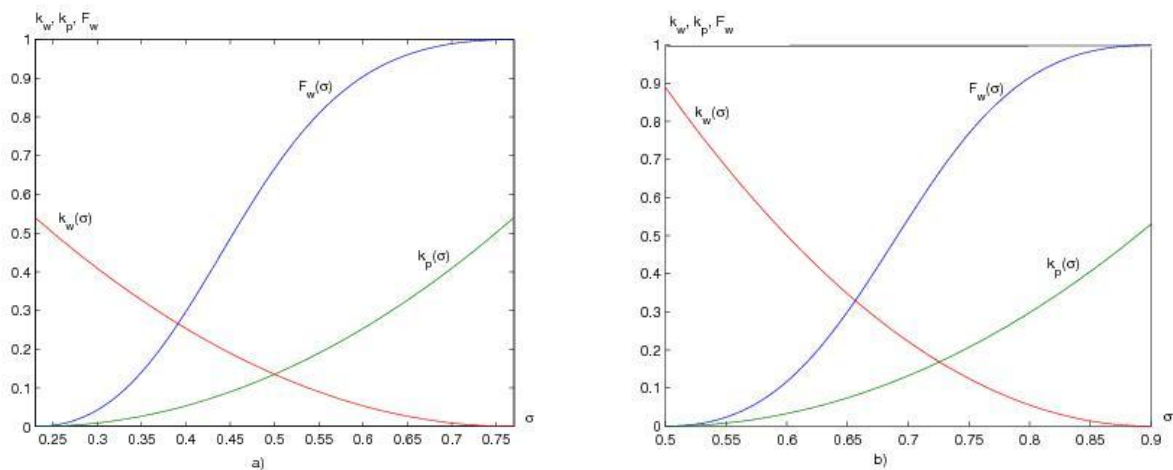
Figure 1: a) The graphs of the functions $k_w(\sigma)$, $k_p(\sigma)$ and $F_w(\sigma)$ corresponding to the experiment 1.; **b)** The graphs of the functions $k_w(\sigma)$, $k_p(\sigma)$ and $F_w(\sigma)$ corresponding to the experiment 2.

According to Buckley-Leverett model, this experiment is modeled by equation (8) with following initial and boundary conditions, [3], [11]

$$\sigma(x,0) = \sigma_0 = 0.23,$$

$$\sigma(0,t) = \sigma_1 = 0.77.$$

Taking into to consideration (65) the functions $k_p(\sigma)$ and $k_w(\sigma)$ are obtained as

$$k_p(\sigma) = \frac{(\sigma - 0.77)^2}{0.54}, \quad k_w(\sigma) = \frac{(\sigma - 0.23)^2}{0.54}.$$

The graphs of the functions $k_p(\sigma), k_w(\sigma)$ and $F_w(\sigma)$ are demonstrated in Fig. 1a. The graphs of the functions obtained by the formulas (30) and (40) are demonstrated in Fig. 2a and 2b. Using the solution of the auxiliary problem (32)-(34) for any t, the front of the displacement of oil by water is found. Since
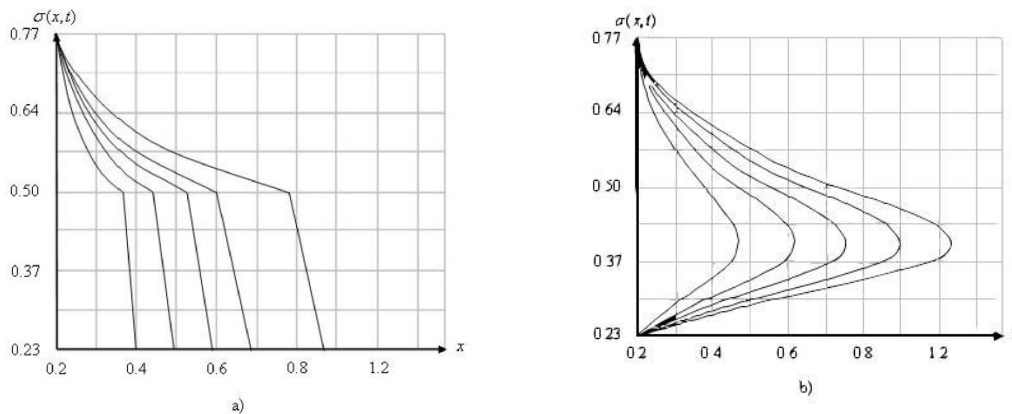
$$\sigma_f = 0.406 \text{ and } F_w(\sigma_f) = 0.69$$

the theoretical degree of efficiency in waterless and wateriness periods are evaluated as

$$\eta_{waterless} = \frac{\sigma_f - \sigma_0}{1 - \sigma_0} \cdot \frac{1 - F_w'(\sigma_0)}{F_w(\sigma_0') - F_w(\sigma_0)} = 0.331 = 33,$$

$$\eta_{total} = \frac{(\bar{\sigma} - \sigma_0) + \dfrac{1 - F_w(\bar{\sigma})}{F_w'(\bar{\sigma})}}{1 - \sigma_0} = 0.51,$$

where $\bar{\sigma}$ is value of saturation on the end of the model.

Figure 2: a) weak solution; **b)** The graph of the function $\sigma(x,t)$ at different values of t, (soft solution)

The second experiment was carried out at room temperature $T = 20^o$ with following physical and chemical data. The length and cross section of the pipe are given as $l = 1.16m$, $S = 0.15m^2$, respectively. The porosity (m) is 0.18, $\bar{\mu} = \dfrac{\mu_w}{\mu_p} = 0.5$. The residual water saturation in this model is $\sigma_0 = 0.5$. The flow rate is

$w = \dfrac{Q}{s} = 0.000002$. The duration of the experiment is 54 hours.

The relative phase permeabilities and the Buckley-Leverett function are given, as follows

$$k_p(s) = \frac{(0.9 - s)^2}{0.18}, \quad k_w(s) = (0.5 - 0.0018T)\frac{(s - 0.5)}{0.14}, \quad F_w(s) = \frac{k_w(s)}{k_w(s) + \bar{\mu}k_p(s)}.$$

The graphs of the functions $k_p(s)$, $k_w(s)$ and $F_w(s)$ are given in Fig. 1b. Like in first experiment this experiment is modeled by equation (8) with following initial and boundary conditions

$$\sigma(x,0) = 0.5, \quad \sigma(0,t) = 0.9.$$

Unlike in the first experiment, the solution of the problem in this case in question is found by the numerical algorithms (53). In Fig.3, the distributions of dynamical saturation of water during the whole experimentation time are given. As it is shown in Fig.3, the time of complete displacement of water is approximately 54 hours. Judging from Fig.3, it is possible to claim that the results obtained from the theoretical problem and the experimental model match quite well.
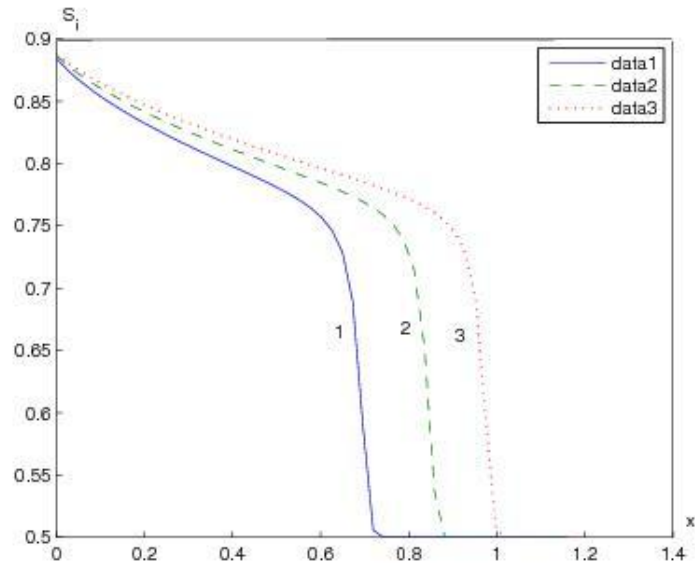
Figure 3: Time evaluation of the water saturations: 1) $T = 10^5$ sec. 2) $T = 1.5 \cdot 10^5$ sec. 3) $T = 2 \cdot 10^5$ sec.

Principally, such a study is attempted as a pre-study for a further project of optimal exploitation of petrol and gas reservoirs.

6. CONCLUSION

In this study, an original method for obtaining the exact and the numerical solutions of the initial and initial-boundary value problems for one-dimensional nonlinear partial differential equations in a class of discontinuous functions is suggested. It is known that the solution describing the process of displacement of oil by water in a porous medium has the shock points, locations of which are unknown beforehand. As to be forced to work with discontinuous functions and to be able to investigate the actual nature of the physical phenomena, it is required to obtain the solution of the studied problem in a class of discontinuous functions. This is why, in this paper, a special method for finding the exact and numerical solutions is suggested.

The obtained results are as follows:

The exact solution of the initial value problem with a non-convex state function is obtained when the initial distribution is a continuous or a piecewise continuous function.

An original method for finding the jump which appears in the solution is developed and its time evaluation is studied.

The higher sensitive differences scheme whose solution accurately expresses all the properties of the physical problem is suggested.

The numerical solution of the Buckley-Leverett problem which describes the macroscopic flow of the two-phase fluid in a porous medium is obtained. Two laboratory models are carried out. In order to model these experiments, it is necessary to find the functions that describe the relative permeability of oil and water phases. The theoretical method for obtaining these features is suggested.

The suggested method allows us to investigate the oil-water interface throughout the process of displacement.

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