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On A Type *A* Semigroup of Congruence Classes

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ABSTRACT---- ***A congruence, characterized by J^*-relations, is constructed on a regular type A semigroup. The resulting set of congruence classes is shown to be a type A semigroup. Commutativity of the morphisms between the semigroups, described by their kernels, is established.***

1. INTRODUCTION

A congruence ρ on a semigroup S is a compatible equivalence on S. The quotient S/ρ can be given a semigroup structure in a natural way and the map $\rho^{\natural}: S \to S/\rho$ defined by $x\rho^{\natural} = x\rho \quad (x \in S)$ is a morphism. ρ is called idempotent – separating if each ρ-class cantains at most one idempotent. ρ is called a group congruence if S/ρ is a group. We, in this piece of article, zero in on the case where S is a type A semigroup.

By constructing a semigroup T consisting of one – one maps between certain left ideals in a type A semigroup S, proving T to be a type A semigroup and then providing a representation of S by T, Asibong – Ibe [1] showed that a representation exists for a type A semigroup similar to Vagner – Preston's representation on inverse semigroups. The result of this work (Asibong-Ibe's work in [1]) is basically the stem of our own result here.

Here, we construct a congruence and then show that its quotient set is a type A semigroup. We then marry up Asibong's representation in [1] with our construction to produce commuting isomorphisms.

2. PRELIMINARIES

Let S be a semigroup and $a, b \in S$. Then, $(a, b) \in L^*$ if aLb in an oversemigroup of S. Thus, by this definition, L^* contains the Green's relation L on S. In an alternative characterisation, Lawson in [7] gave that for $a, b \in S$, $(a, b) \in L^*$ if $\forall x, y \in S^1, ax = ay$ if and if $bx = by$.

Lemma 2.1: Let S be a semigroup and e an idempotent in S. Then, $\forall a \in S$, the following are equivalent:

i) $(e, a) \in L^*$ and (ii) $\forall x, y \in S, ax = ay$ if and if $ex = ey$.

R^* is dual to L^* and the above definition of L^* apply in dual manner to R^*.

The intersection of L^* and R^* is denoted by H^*. The join of L^* and R^* on S is the equivalence D^*. In general, $L^* \text{o}\, R^* \neq R^* \text{o}\, L^*$ and neither equals D^*. Basically, aD^*b if and only if there exists elements $x_1, x_2, x_3, \dots, x_n$ in S such that $aL^*x_1R^*x_2L^*x_3 \dots x_{n-1}L^*x_nR^*b$.

$D \subseteq D^*$ and $H \subseteq H^*$. If S is regular, then $L = L^*$ and $R = R^*$.

Let S be a semigroup and I an ideal of S. Then I is called *-ideal if $L_a^* \subseteq I$ and $R_a^* \subseteq I$ for all $a \in I$. The smallest *-ideal containing a is the principal *-ideal generated by a and is denoted by $J^*(a)$. For $a, b \in S$, aJ^*b if and only if $J^*(a) = J^*(b)$. The J^*-class containing the element $a \in S$ is denoted by J_a^*

From Lawson in [7], we note that L^* is a right congruence and R^* is a left congruence. Thus, J^* is a congruence and by congruence property, we have –

$J_a^* \cdot J_a^* = J_{ab}^*$, $(J_a^*)^2 = J_{a^2}^*$, $J_{ab}^* \leq J_a^*$, $J_{ab}^* \leq J_b^*$. The relation J^* contains D^*.

A semigroup S is said to be $*$-simple if all its elements are J^* related and *-bisimple semigroup if it contains one D^*-class.

A semigroup S is said to be an $abundant$ semigroup if each L^*-class and each R^*-class contains an idempotent and it is $superabundant$if each H^*-class contains an idempotent.

An abundant semigroup S whose idempotents form a semilattice $E(S)$ is called $adequate$. In an abundant semigroup, the idempotents in each L^*-class and each R^*-class are unique. If S is adequate, and a is an element of S, then $a^*(a^+)$ will denote the unique idempotent in the L^*-(R^*-)class of a . Thus, in an adequate semigroup, $aL^*b \Leftrightarrow a^* = b^*$ and $aR^*b \Leftrightarrow a^+ = b^+$

An adequate semigroup S is said to be a $type\ A$ semigroup if for each a in S and e in $E(S)$, $ea = a(ea)^*$ and $ae = (ae)^+a$.

Fountain in [4] characterised a type A semigroup as follows:

Lemma 2.2: Let S be an adequate semigroup. Then, $\forall a \in S$ and $\forall e \in E(S)$, the following are equivalent:

(i) S is a type A semigroup.
(ii) $eS^1 \cap aS^1 = eaS^1$ and $S^1e \cap S^1a = S^1ae$ and
(iii) There exist embeddings $\lambda_1: S \to S_1$ and $\lambda_2: S \to S_2$ into inverse semigroup S_1, S_2 such that $a^*\lambda_1 = (a\lambda_1)^{-1}(a\lambda_1)$ and $a^+\lambda_2 = (a\lambda_2)(a\lambda_2)^{-1}$.

A type A semigroup is called a *-bisimple semigroup if it contains precisely one D^*-class and one regular D-class.

Let S be a type A semigroup with $a, b \in S$. The relation $\widetilde{D}$ is defined on S by $(a, b) \in \widetilde{D}$ if and only if $(a^*, b^*) \in D$ and $(a^+, b^+) \in D$ for some a^*, b^* a^+ and b^+. $\widetilde{D}$ is an equivalence relation and the inclusion- $D \subseteq \widetilde{D} \subseteq D^*$ holds.

Asibong – Ibe [2] showed that for an adequate semigroup S, D^* and $\widetilde{D}$ coincide if and only if every nonempty H^*-class contains a regular element. The equality - $D^* = \widetilde{D}$, guarantees the equality - $D^* = L^* o R^* = R^* o L^*$.

A semigroup homomorphism $\rho: S \to T$ is said to be a $good$ homomorphism if for all $a, b \in S$, $a\, L^*(S) b$ implies $a\rho\, L^*(T) b\rho$ and that $a\, R^*(S) b$ implies $a\rho\, R^*(T) b\rho$.

A congruence δ on a semigroup S is said to be a $good$ congruence if the natural homomorphism from S onto S/δ is $good$.

The following lemmas are adapted from El-Qallali in [3] :

Lemma 2.3: Let S be an abundant semigroup and $\rho: S \to T$ a semigroup homomorphism. Then the following statements are equivalent:

i. The homomorphism ρ is good

ii. For each element $a \in S$, there are idempotents $e,\ f$, with $e \in L_a^*,\ \ f \in R_a^*$ such that $a\rho L^*(T)e\rho$ and $a\rho R^*(T)f\rho$

Lemma 2.4: Let ρ be a congruence on an abundant semigroup S. Then the following statements are equivalent:

i. ρ is a good congruence
ii. For all $a \in S$, there are idempotents $e,\ f$, with $e \in L_a^*,\ \ f \in R_a^*$ such that $a\rho L^*(S/\rho)e\rho$ and $a\rho R^*(S/\rho)f\rho$

It therefore implies that a congruence ρ on an abundant semigroup S is good if $\forall a \in S$ and $\forall x, y \in S^1$ there are idempotents $e,\ f$, with $eL^*a,\ \ fR^*a$ such that $(ax, ay) \in \rho$ implies $(ex, ey) \in \rho$ and $(xa, ya) \in \rho$ implies $(xf, yf) \in \rho$. Corresponding interpretation also goes to a good homomorphism on an abundant semigroup.

In general, the homomorphic image of an abundant semigroup is not abundant. We however can quote from [8] that the good homomorphic image of an abundant semigroup is always abundant. The following lemma comes from [8]

Lemma 1.5: The intersection of good congruences is a congruence.

Proof: Let ρ, σ be good congruences and suppose $a \in S$ and that $(ax, ay) \in \rho \cap \sigma$ for all $x, y \in S^1$. Then $(ax, ay) \in \rho$ and $(ax, ay) \in \sigma$ and therefore for some $e_1, e_2 \in L_a^* \cap E(S)$, $(e_1x, e_1y) \in \rho$ and $(e_2x, e_2y) \in \sigma$. Now, for some $e \in L_a^* \cap E(S)$, $(ee_1x, ee_1y) \in \rho$ and $(ee_2x, ee_2y) \in \sigma$. Since e_1, e_2 are right identities in L_a^*, we have $(ex, ey) \in \rho \cap \sigma$. Similarly, $[(xa, ya) \in \rho \cap \sigma] \Rightarrow [(xf, yf) \in \rho \cap \sigma]$ for some $f \in R_a^* \cap E(S)$.

We conclude the section with the following definitions:

A semigroup homomorphism $\varphi: S \to T$ is said to be a *-*homomorphism* if for all $a, b \in S$, $a\,L^*(S)b$ if and only if $a\varphi\ L^*(T)b\varphi$ and $a\ R^*(S)b$ if and only if $a\varphi\ R^*(T)b\varphi$.

A congruence δ on a semigroup S is said to be a *-*congruence* if the natural homomorphism from S onto S/δ is a *-*homomorphism*.

3. A CONGRUENCE ON A TYPE A SEMIGROUP

In this and subsequent sections, the term semigroup S will refer to a regular type A semigroup S with $E(S)$ as its set of idempotents. We recall that a semigroup S is called *regular* if for all $a \in S$ there exists $x \in S$ such that $axa = a$. Now, for $a \in S$, $a^+, a^* \in E(S)$, $a^+ = aa^{-1}$, $a^* = a^{-1}a$ and $aa^* = a^+a = a$.

Lemma 3.1: For all $a, b \in S$, the following statements are true:

i)	$a^*b^+ = (ab^+)^*$	iii)	$a^{++} = a^+$	v)	$a^*b = b(ab)^*$
ii)	$a(ab^+)^* = (ab^+)^+a$	iv)	$(ab^+)^+ = (ab)^+$	vi)	$(ab)^* = (a^*b)^*$

Let S be a type A semigroup S and $E(S)$ its semilattice of idempotents. Now let the J^*-class containing an element $e \in E(S)$ be denoted by $E(e)$. For $a, b \in S$, define a relation δ on S by $(a, b) \in \delta$ if and only if $b = eaf$ and $a = gbh$ for some $e \in E(a^+)$, $f \in E(a^*)$, $g \in E(b^+)$ and $h \in E(b^*)$.

Lemma 3.2: Then, δ is a congruence on S.

Proof: We start by showing that δ is an equivalence.

$(a,a) \in \delta$ since $a^+aa^* = aa^* = a$ for $a^+ \in E(a^+)$ and $a^* \in E(a^*)$. Thus, δ is reflexive.

By definition, δ is symmetric. For transitivity, let $(a,b) \in \delta$ and $(b,c) \in \delta$ with $a,b,c \in S$. Therefore, for some $e_1 \in E(a^+)$, $f_1 \in E(a^*)$, $g_1, g_2 \in E(b^+)$, $h_1, h_2 \in E(b^*)$, $e_2 \in E(c^+)$ and $f_2 \in E(c^*)$,

$$b = e_1 a f_1 \text{ and } a = g_1 b h_1 \quad ; \quad b = e_2 c f_2 \text{ and } c = g_2 b h_2$$

So that $a = g_1 e_2 c f_2 h_1$ and $c = g_2 e_1 a f_1 h_2$

With $g_1 e_2 \in E(b^+c^+) = E(c^+b^+) \subseteq E(c^+)$ and $f_2 h_1 \in E(c^*b^*) = E(b^*c^*) \subseteq E(c^*)$;

$g_2 e_1 \in E(b^+a^+) = E(a^+b^+) \subseteq E(a^+)$ and $f_1 h_2 \in E(a^*b^*) = E(b^*a^*) \subseteq E(a^*)$.

Hence, $(a,c) \in \delta$, which establishes transitivity of δ.

Now, for compatibility of δ, assume $(a,b) \in \delta$ so that $b = eaf$ and $a = gbh$ for some $e \in E(a^+)$, $f \in E(a^*)$, $g \in E(b^+)$ and $h \in E(b^*)$. For any $c \in S$, $bc = eafc$.

If we choose f to be equal to a^*, then $bc = eaa^*c = eac(ac)^* = e(ac)^+ac(ac)^*$.

We recall that each $E(e)$, $[e \in E(S)]$, is a J^*-class and therefore a congruence class. So that

$$e(ac)^+ \in E(a^+).E(ac)^+ = E(a^+)(ac)^+ = E(ac)^+(a^+) \subseteq E(ac)^+.$$

And if we choose h to be equal to b^*,

$$ac = gbb^*c = gbc(bc)^* = g(bc)^+bc(bc)^*, \; g(bc)^+ \in E(bc)^+.$$

Therefore, $(ac, bc) \in \delta$. Thus, δ is right compatible. Proof of left compatibility of δ comes in a similar fashion. We therefore conclude that δ is a congruence.

Proposition 3.3: δ is good on S.

Proof: For $a,x,y \in S$, let $(ax, ay) \in \delta$. This implies that $ay = eaxf$ and $ax = gayh$ for some $e \in E(ax)^+$, $f \in E(ax)^*$, $g \in E(ay)^+$ and $h \in E(ay)^*$.

$ay = eaxf \Rightarrow a^{-1}ay = a^{-1}eaxf$. If we choose $e = (ax)^+$, then we have

$$a^{-1}ay = a^{-1}(ax)^+axf = a^{-1}axf = (a^{-1}ax)^+a^{-1}axf.$$

$a^{-1}a \in L_a^*$, and $f \in E(ax)^* = E(aa^{-1}ax)^* \subseteq E(a^{-1}ax)^*$

Now, $ax = gayh \Rightarrow a^{-1}ax = a^{-1}gayh$.

Taking $g = (ay)^+$, we have $a^{-1}ax = a^{-1}(ay)^+ayh = a^{-1}ayf = (a^{-1}ay)^+a^{-1}ayh.$

$$h \in E(ay)^* = E(aa^{-1}ay)^* \subseteq E(a^{-1}ay)^*$$

We have just shown that for all $a,x,y \in S$, there exists $u = a^{-1}a \in L_a^*$ such that $[(ax,ay) \in \delta] \Rightarrow [(ux,uy) \in \delta]$.

In a similar approach, it can be shown that $[(xa,ya) \in \delta] \Rightarrow [(xv,yv) \in \delta]$ with $v \in R_a^*$.

Thus, δ is good.

Having established that δ is a congruence, the very natural next step is to define a binary operation on the quotient set S/δ which is the set of congruence classes of δ. We define the operation in a natural way as follows: $(a\delta)(b\delta) = (ab)\delta$

Compatibility of δ makes it possible and easy to see that our operation here is well-defined. We notice that for all $a, b, c, d \in S$,

$$a\delta = c\delta \text{ and } b\delta = d\delta \quad \Rightarrow (a, c) \in \delta \text{ and } (b, d) \in \delta \Rightarrow (ab, cd) \in \delta \quad \Rightarrow (ab)\delta = (cd)\delta.$$

Obviously the operation is associative, and therefore S/δ is a semigroup.

Theorem 3.4: S/δ is a type A semigroup.

We establish the proof through the following lemmas:

Lemma 3.5 For all $a, b \in S$,

i. $(a\delta, b\delta) \in L^*(S/\delta)$ if and only if $(a, b) \in L^*(S)$ and
ii. $(a\delta, b\delta) \in R^*(S/\delta)$ if and only if $(a, b) \in R^*(S)$

Proof: Assume $(a\delta, b\delta) \in L^*(S/\delta)$. This implies that for all $c\delta, d\delta \in S/\delta$ (which implies $\forall c, d \in S$)

$a\delta . c\delta = a\delta . d\delta$ if and only if $b\delta . c\delta = b\delta . d\delta$. That is $ac\delta = ad\delta$ if and only if $bc\delta = bd\delta$

Now, $ac\delta = ad\delta$ means $(ac, ad) \in \delta$ and this implies that for some $e \in E(ac)^+$, $f \in E(ac)^*$, $g \in E(ad)^+$ and $h \in E(ad)^*$, $ad = eacf$ and $ac = gadh$

Choosing $e = (ac)^+$ and $f = (ac)^*$, then we have $ad = (ac)^+ac(ac)^* = ac(ac)^* = ac$

Choosing $g = (ad)^+$ and $f = (ad)^*$ will also produce $ac = ad$.

Similarly, taking up $bc\delta = bd\delta$ will produce $bc = bd$. Therefore, $(a, b) \in L^*(S)$.

Conversely, let $(a, b) \in L^*(S)$. Then for all $c, d \in S$, $ac = ad$ and $bc = bd$.

Since ac, ad, bc and bd are all in S, $ac\delta, ad\delta, bc\delta$ and $bd\delta$ are all in S/δ.

With $ac = ad$ and $bc = bd$, we have $ac\delta = ad\delta$ and $bc\delta = bd\delta$.

That is $a\delta c\delta = a\delta d\delta$ and $b\delta c\delta = b\delta d\delta$ for all $c\delta, d\delta \in S/\delta$

Thus, $(a\delta, b\delta) \in L^*(S/\delta)$.

Proof of (ii) is similar.

The following corollary is consequent upon the right above lemma.

Corollary 3.6 Let $a\delta, b\delta \in S/\delta$, then

i. $(a\delta, b\delta) \in H^*(S/\delta)$ if and only if $(a, b) \in H^*(S)$ and
ii. $(a\delta, b\delta) \in D^*(S/\delta)$ if and only if $(a, b) \in D^*(S)$

Proof: (i) $[(a\delta, b\delta) \in H^*(S/\delta)] \Leftrightarrow [(a\delta, b\delta) \in L^*(S/\delta)$ and $(a\delta, b\delta) \in R^*(S/\delta)]$

$$\Leftrightarrow [(a, b) \in L^*(S) \text{ and } (a, b) \in R^*(S)] \quad \Leftrightarrow \quad (a, b) \in H^*(S).$$

(ii) For some $c_1\delta,\ c_2\delta,\ c_3\delta,\ \dots,\ c_n\delta\ \in S/\delta$,

$$[(a\delta, b\delta) \in D^*(S/\delta)] \Leftrightarrow [a\delta L^*(S/\delta)c_1\delta R^*(S/\delta)c_2\delta L^*(S/\delta)c_3\delta \dots c_n\delta R^*(S/\delta)b\delta]$$

$$\Leftrightarrow [aL^*(S)c_1R^*(S)c_2L^*(S)c_3 \dots c_nR^*(S)b] \Leftrightarrow (a,b) \in D^*(S).$$

Lemma 3.7 An element $a\delta \in S/\delta$ is an idempotent if and only if $a \in S$ is an idempotent.

$E(S/\delta)$, the set of idempotents of S/δ, is a semilattice.

Proof: Suppose $a\delta$ is idempotent in S/δ. It means that $(a\delta)^2 = a^2\delta = a\delta$. That is $(a^2, a) \in \delta$.

So that for some $\in E(a^2)^+$, $f \in E(a^2)^*$, $g \in E(a)^+$ and $h \in E(a)^*$, $a = ea^2f$ and $a^2 = gah$.

Choosing $g = e$ and $h = af$ guarantees $a = a^2$. And $g = e$ and $h = af$ are well – chosen since $e \in E(a^2)^+ \subseteq E(a)^+$ and $af \in E(a)^*$. $E(a^2)^* = E(a)^*(a^2)^* = E(a^2)^*(a)^* \subseteq E(a)^*$.

Conversely, $a^2 = a$ implies that $a^2\delta = a\delta$. That is $(a\delta)^2 = a\delta$.

Now, assume $e\delta, f\delta \in E(S/\delta)$. Then $e, f \in E(S)$ and therefore $(e\delta)(f\delta) = ef\delta = fe\delta = (f\delta)(e\delta)$

And if $e \leq f$, $ef = fe = e$, and so $e\delta f\delta = f\delta e\delta = e\delta$. Thus, $E(S/\delta)$ is a semilattice.

For $a \in S$, $a^* \in L_a^*$, $a^+ \in R_a^*$ and $a\delta a^*\delta = aa^*\delta = a\delta$, $a^+\delta a\delta = a^+a\delta = a\delta$. So, we evidently have the following facts:

Lemma 3.8 For each $a\delta \in S/\delta$, $(a\delta, a^*\delta) \in L^*(S/\delta)$ and $(a\delta, a^+\delta) \in R^*(S/\delta)$.

Furthermore, let $L_{a\delta}^*$ and $R_{a\delta}^*$ be, respectively, the $L^*(S/\delta)$ and $R^*(S/\delta)$ classes containing $a\delta$. Let us denote by $a\delta^*$ and $a\delta^+$ the unique idempotents in $L_{a\delta}^*$ and $R_{a\delta}^*$ respectively.
Now, for $a \in S$ and $e \in E(S)$, $ea = a(ea)^*$ and $ae = (ae)^+a$.
Consequently, $e\delta a\delta = ea\delta = a(ea)^*\delta = [a\delta][(ea)^*\delta] = [a\delta][(ea\delta)^*]$
$= [a\delta][(e\delta a\delta)^*] = a\delta(e\delta a\delta)^*$
Similarly, $a\delta e\delta = (a\delta e\delta)^+ a\delta$. Thus, we have shown that

Lemma 3.9 For each $a\delta, e\delta \in S/\delta$, $e\delta a\delta = a\delta(e\delta a\delta)^*$ and $a\delta e\delta = (a\delta e\delta)^+ a\delta$.

All the lemmas 2.5 to 2.9 and the observations therein make the proof of theorem 2.4.

4. THE ISOMORPHISMS

Asibong in [1] established that there is a Vagner – Preston type representation from a type A semigroup S into a type A semigroup T of mappings, where $T = \{\alpha_a \mid a \in S,\ \alpha_a\colon Sa^+ \to Sa^*\}$. It was, thus, shown that that the mapping $\varphi\colon S \to T$ with $a\varphi = \alpha_a$ is an isomorphism from S onto T. It follows from the general definition given by Howie in [6] that

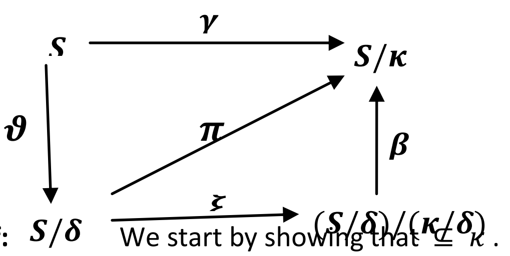
$$\text{Ker}\varphi = \varphi\ o\ \varphi^{-1} = \{(a,b) \in S \times S\colon a\varphi = b\varphi\}$$

Kerφ is obviously an equivalence relation on S. It is not just an equivalence, it is a congruence on S. To see this, let $(a,b), (x,y) \in$ Kerφ. This implies that $a\varphi = b\varphi$ and $x\varphi = y\varphi$. Therefore, $ax\varphi = a\varphi x\varphi = b\varphi y\varphi = by\varphi$. So that $(ax, by) \in$ Kerφ.

We know, from our elementary algebra, that there should be a natural morphism γ (say) from S onto $S/$Kerφ defined by $a\gamma = a$Kerφ, $(a \in S)$. Similarly, we have a natural morphism $\vartheta\colon S \to S/\delta$ defined $a\vartheta = a\delta$, $(a \in S)$.

For convenience, let us, for the rest of this section, denote Kerφ by κ. The last paragraph is part of the following theorem:

Theorem 3.1 $\delta \subseteq \kappa$. There is a isomorphism π from S/δ onto S/κ whose kernel – κ/δ, is a congruence on S/δ such that $(S/\delta)/(\kappa/\delta)$ is isomorphic to S/κ and such that the diagram



commutes.

Proof: We start by showing that $\delta \subseteq \kappa$.

Suppose $(a,b) \in \delta$, $a,b \in S$. This implies that for some $e \in E(a^+)$, $f \in E(a^*)$, $g \in E(b^+)$ and $h \in E(b^*)$, $b = eaf$ and $a = gbh$.

So that $a\varphi = gbh\varphi = geafh\varphi = g\varphi e\varphi a\varphi f\varphi h\varphi = (ge)\varphi \,.\, a\varphi \,.\, (fh)\varphi = (ge)a(fh)\varphi$.

Now, $ge \in E(b^+) \,.\, E(a^+) = E(b^+)(a^+) = E(a^+)(b^+) \subseteq E(a^+)$

and $fh \in E(a^*) \,.\, E(b^*) = E(a^*)(b^*) = E(b^*)(a^*) \subseteq E(a^*)$.

Therefore, $a\varphi = (ge)a(fh)\varphi = b\varphi$. Thus, $(a,b) \in \kappa$.

Next,

Define a map $\pi : S/\delta \to S/\kappa$ by $(a\delta)\pi = a\kappa$ with $a \in S$. π is well defined since $[a\delta = b\delta] \Rightarrow [\,(a,b) \in \delta\,] \Rightarrow [\,(a,b) \in \kappa\,] \Rightarrow [\,a\kappa = b\kappa\,] \Rightarrow [\,(a\delta)\pi = (b\delta)\pi\,]$.

π is a morphism since $(a\delta b\delta)\pi = (ab\delta)\pi = (ab)\kappa = a\kappa b\kappa = (a\delta)\pi\,(b\delta)\pi$.

Now, suppose $a\kappa = b\kappa$. This implies that $a\varphi = b\varphi$ and therefore $\alpha_a = \alpha_b$, which in turn implies that $Sa^+ = Sb^+$, $Sa = Sb$, the domains and ranges of α_a and α_b respectively. $Sa^+ = Sb^+$ means that $Ea^+ = Eb^+$, $Sa = Sb$ also means that $Ea = Eb$ and evidently, $Ea^* = Eb^*$. Thus, with $a \in Ea^+$, we have $a \in Eb^+$. Similarly, $a \in Eb^*$. So that there is some $g \in Eb^+$ and some $h \in Eb^+$ such that $a = gbh$. In the same vein, $b \in Ea^+$ and $b \in Ea^*$ and for some $e \in Ea^+$ and $f \in Ea^*$, $b = eaf$. Hence, $a\delta = b\delta$. That is, π is one – one.

The definition of π makes it obviously surjective since for all $a \in S$, every $a\kappa$ corresponds to $a\delta$. Thus, π is an isomorphism.

The kernel of π is defined as follows:

$\ker \pi = \pi \circ \pi^{-1} = \{(a\delta, b\delta) \in S/\delta \times S/\delta : (a\delta)\pi = (b\delta)\pi\} = \{\,(a\delta, b\delta) \in S/\delta \times S/\delta : a\kappa = b\kappa\,\}$.

We can therefore denote the kernel of π as κ/δ and then write

$$\kappa/\delta = \{\,(a\delta, b\delta) \in S/\delta \times S/\delta : (a,b) \in \kappa\,\}.$$

κ/δ is clearly an equivalence on S/δ. To show that it is a congruence on S/δ, assume $(a\delta, b\delta), (c\delta, d\delta) \in \kappa/\delta$. This implies that $(a,b), (c,d) \in \kappa$, and therefore

$a\kappa \,.\, ck = b\kappa \,.\, d\kappa$. So that $ac\kappa = bd\kappa$. Thence, $(ac, bd) \in \kappa$. This implies that

$(ac\delta, bd\delta) = (a\delta c\delta, b\delta d\delta) \in \kappa/\delta$, with $(a\delta c\delta, b\delta d\delta \in S/\delta)$.

As usual, there is therefore a natural morphism $\xi : S/\delta \to (S/\delta)/(\kappa/\delta)$ defined by

$(a\delta)\xi = (a\delta)(\kappa/\delta)$ where $a\delta \in S/\delta$.

Now, define the map $\beta: (S/\delta)/(\kappa/\delta) \to S/\kappa$ by $[(a\delta)(\kappa/\delta)]\beta = a\kappa$.

To show that β is well defined, let us suppose that $a\delta(\kappa/\delta) = b\delta(\kappa/\delta)$.

$[a\delta(\kappa/\delta) = b\delta(\kappa/\delta)] \Rightarrow [(a\delta, b\delta) \in \kappa/\delta] \Rightarrow [(a,b) \in \kappa] \Rightarrow [a\kappa = b\kappa]$.

Having ascertained that β is well defined, we shall now show that it is a morphism.

$$[(a\delta)(\kappa/\delta).(b\delta)(\kappa/\delta)]\beta = [a\delta b\delta(\kappa/\delta)]\beta = [ab\delta(\kappa/\delta)]\beta$$

$$= ab\kappa = a\kappa b\kappa = [(a\delta)(\kappa/\delta)]\beta\,[(b\delta)(\kappa/\delta)]\beta.$$

Thus, β is a morphism.

Our next goal is to show that β is one – one. And to do that, assume $a\kappa = b\kappa$. So that $(a,b) \in \kappa$, which guarantees that $(a\delta, b\delta) \in \kappa/\delta$. And therefore, $a\delta(\kappa/\delta) = b\delta(\kappa/\delta)$ as required.

By the definition of β, for all $a \in S$, every $a\kappa$ in S/κ has $[(a\delta)(\kappa/\delta)]$ in $(S/\delta)/(\kappa/\delta)$ assigned to it. So, evidently, β is surjective. β is therefore an isomorphism.

Finally, we notice that

$(a)\vartheta\pi = (a\delta)\pi = a\kappa$, $a\gamma = a\kappa$. Therefore $\vartheta\pi = \gamma$.

$(a\delta)\xi\beta = [(a\delta)(\kappa/\delta)]\beta = a\kappa$. Therefore $\xi\beta = \pi$.

Thus, $\vartheta\xi\beta = \vartheta\pi = \gamma$. Hence the diagram commutes.

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6. REFERENCES

[1] Asibong-Ibe U : *Representation of Type A Monoids*. Bull Austral Math Soc. 44(1991) 131 – 138.

[2] Asibong-Ibe U: *$*$-Simple Type A ω-Semigroups.* Semigroup Forum 47 (1993) 135 – 149.

[3] El-Qallali : *Quasi – Adeqaute Semigroups.* International Center for Theoretical Physics, Trieste – Italy (1987).

[4] Fountain J. B: *Adequate Semigroups.* Proc. Edinburgh Math. Soc. 22 (1979) 113 – 125.

[5] Howie J. M: *Fundamentals of Semigroup Theory.* Oxford University Press Inc. (1995)

[6] Howie J. M: The Maximum Idempotent – Separating Congruence on an Inverse Semigroup. Glasgow University (1963).

[7] Lawson M. V : *The Structure of Type A Semigroups*. Quart. J, Math. Oxford (2), 37 (1986), 279 – 298.

[8] Ren X. M., Shum K. P: *The Structure of Q^*-Inverse Semigroups.* Journal of Algebra 325 (2011) 1 – 17.