The Properties of Generalized k-Pell like Sequence using Matrices

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ABSTRACT— ***The Pell sequence has been generalized in many ways. In this study, we define new generalization $\{M_{k,n}\}$ with initial conditions $M_{k,0}=4,\ M_{k,1}=m+4$, which is generated by the requrrence relation $M_{k,n+1}=kM_{k,n}+M_{k,n-1}$ for $n\geq 1$, where k,m are integer numbers. Then, we obtain some properties related to new generalization of Pell sequence.***

Keywords— Pell sequence, recurrence relation

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1. INTRODUCTION

The well-known Pell $\{P_n\}$ and Pell-Lucas $\{Q_n\}$ sequences have many interesting properties and their applications to every fields of positive science and art [1-2]. They are defined for $n\geq 2$ with the recurrences $P_n=2P_{n-1}+P_{n-2}$, $(P_0=0,\ P_1=1)$ and $Q_n=2Q_{n-1}+Q_{n-2}$, $(Q_0=2,\ Q_1=2)$ respectively. In the literature, these numbers have been generalized in many ways [1-5]. Falcon and Plaza, in [6], defined the k-Fibonacci sequence $\{F_{k,n}\}_{n=0}^{\infty}$, $k\geq 1,\ n\geq 1$ and k-Lucas sequence $\{L_{k,n}\}_{n=0}^{\infty}$, $k\geq 1,\ n\geq 1$,

$$F_{k,n+1}=kF_{k,n}+F_{k,n-1},\ (F_{k,0}=0,\ F_{k,1}=1)$$

and

$$L_{k,n+1}=kL_{k,n}+L_{k,n-1},\ (L_{k,0}=2,\ L_{k,1}=k)$$

respectively. Many properties of these numbers were deduced directly from elementary matrix algebra. Furthermore the 3-dimensional k-Fibonacci spirals were studied from a geometric points of view. In [3-4], Taskara N., Uslu K., Gulec H. H., gave the binomial properties Fibonacci and Lucas sequences and obtained some new algebraic results of these numbers. In [2], Horadam showed that some properties involving Pell numbers. Horadam gave Simpson formula

$$P_{n+1}P_{n-1}-P_n^2=(-1)^n$$

for the Pell numbers. In [7], Godase A. D. defined generalized k-Fibonacci like sequence using matrices, studied some properties of these numbers.

2. THE GENERALIZED k - PELL LIKE SEQUENCE

By using [7], we defined a new generalization of the k -Pell sequences and gave few terms of this sequence.

Definition 2.1. For any integer number $k \geq 1$ and $m \geq 0$ the generalized k -Pell like sequence $M_{k,n}$ is defined by

$$M_{k,n+1} = 2M_{k,n} + kM_{k,n-1},\ (n \geq 1),\ (M_{k,0} = 4,\ M_{k,1} = m+4).$$

Characteristics equation of the initial recurrence relation is $r^2 - 2r - k = 0,$ and characteristics roots are

$$r_1 = 1 + \sqrt{1+k}\ ,\quad r_2 = 1 - \sqrt{1+k}\ .$$

Characteristics roots verify the properties

$$r_1 - r_2 = 2\sqrt{1+k}\ ,\ r_1 + r_2 = 2,\ \ r_1 r_2 = -k.$$

It is clear from the definition of the generalized k -Pell like sequence it satisfy

$$M_{k,n} = mP_{k,n} + Q_{k,n},\ (n \geq 0) \tag{2.1}$$

where $P_{k,n}$ and $Q_{k,n}$ are k -Pell and k -Pell-Lucas numbers respectively. $P_{k,n}$ and $Q_{k,n}$ are defined by the solutions of the following discrete equalities

$$P_{k,n+1} = 2P_{k,n} + kP_{k,n-1},\ (n \geq 1)$$

$$Q_{k,n+1} = 2Q_{k,n} + kQ_{k,n-1},\ (n \geq 1)$$

with initial conditions $P_{k,0} = 0,\ P_{k,1} = 1$ and $Q_{k,0} = 2,\ Q_{k,1} = 2$, respectively.

First few terms of the generalized k -Pell like sequences are:

$M_{k,0} = 4,$

$M_{k,1} = m + 4,$

$M_{k,2} = 4k + 2m + 8,$

$M_{k,3} = (m+12)k + 4m + 16,$

$M_{k,4} = 4k^2 + (4m+32)k + 8m + 32,$

$M_{k,5} = (20+m)k^2 + (12m+80)k + 16m + 64,$

$M_{k,6} = 4k^3 + (6m+72)k^2 + (32m+192)k + 32m + 128.$

3. PROPERTIES OF GENERALIZED k - PELL LIKE SEQUENCE BY MATRIX METHODS

In this section we give our obtained results related to k-Pell Like sequence.

Theorem 3.1. For the generalized k -Pell like sequence $M_{k,n}$, the follows equality holds

$$\begin{pmatrix} M_{k,n+1} & M_{k,n} \\ M_{k,n} & M_{k,n-1} \end{pmatrix} = L^n \begin{pmatrix} m+4 & 4 \\ 4 & (m-4)/k \end{pmatrix}, \quad \text{where } L = \begin{pmatrix} 2 & 1 \\ k & 0 \end{pmatrix}. \tag{3.1}$$

Proof: Let us use the principle of mathematical induction on n. For $n=1$, it is easy to see that the equality holds

$$\begin{pmatrix} M_{k,2} & M_{k,1} \\ M_{k,1} & M_{k,0} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ k & 0 \end{pmatrix} \begin{pmatrix} m+4 & 4 \\ 4 & (m-4)/k \end{pmatrix} = \begin{pmatrix} 4k+2m+8 & m+4 \\ m+4 & 4 \end{pmatrix}.$$

Now, assume that result is true for $n-1$. Therefore we have

$$\begin{pmatrix} M_{k,n} & M_{k,n-1} \\ M_{k,n-1} & M_{k,n-2} \end{pmatrix} = L^{n-1} \begin{pmatrix} m+4 & 4 \\ 4 & (m-4)/k \end{pmatrix}.$$

Now, if we multiply the matrix L the last equation, then we can write the following equation

$$\begin{pmatrix} M_{k,n} & M_{k,n-1} \\ M_{k,n-1} & M_{k,n-2} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ k & 0 \end{pmatrix} = L^{n-1} \begin{pmatrix} m+4 & 4 \\ 4 & (m-4)/k \end{pmatrix} \begin{pmatrix} 2 & 1 \\ k & 0 \end{pmatrix},$$

$$\begin{pmatrix} M_{k,n+1} & M_{k,n} \\ M_{k,n} & M_{k,n-1} \end{pmatrix} = L^n \begin{pmatrix} m+4 & 4 \\ 4 & (m-4)/k \end{pmatrix}.$$

Theorem 3.2. (Simpson's identity for negative n **)**

$$M_{k,-n+1} M_{k,-n-1} - M_{k,-n}^2 = \left(\frac{m^2 - 16k - 16}{k} \right)$$

Proof: If we get $-n$ instead of n in matrix equation 3.1., then we have

$$\begin{pmatrix} M_{k,-n+1} & M_{k,-n} \\ M_{k,-n} & M_{k,-n-1} \end{pmatrix} = L^{-n} \begin{pmatrix} m+4 & 4 \\ 4 & (m-4)/k \end{pmatrix}.$$

$$L^{-n} = \begin{pmatrix} P_{k,n+1} & kP_{k,n} \\ P_{k,n} & kP_{k,n-1} \end{pmatrix}^{-n} = \frac{1}{k^n (P_{k,n+1}P_{k,n-1} - P_{k,n}^2)^n} \begin{pmatrix} kP_{k,n-1} & -kP_{k,n} \\ -P_{k,n} & P_{k,n+1} \end{pmatrix} = \frac{1}{(-1)^n k^n} \begin{pmatrix} kP_{k,n-1} & -kP_{k,n} \\ -P_{k,n} & P_{k,n+1} \end{pmatrix}$$

From the last equations, we can write

$$\begin{pmatrix} M_{k,-n+1} & M_{k,-n} \\ M_{k,-n} & M_{k,-n-1} \end{pmatrix} = \frac{1}{(-1)^n k^n} \begin{pmatrix} kP_{k,n-1} & -kP_{k,n} \\ -P_{k,n} & P_{k,n+1} \end{pmatrix} \begin{pmatrix} m+4 & 4 \\ 4 & (m-4)/k \end{pmatrix}.$$

If we calculate the determinant of above matrix equation, we have

$$M_{k,-n+1}M_{k,-n-1} - M_{k,-n}^2 = \frac{1}{(-1)^n k^n} \left[P_{k,n+1}P_{k,n-1} - P_{k,n}^2 \right] ((m+4)(m-4) - 16k)$$

$$M_{k,-n+1}M_{k,-n-1} - M_{k,-n}^2 = \frac{1}{(-1)^n k^n} \left[k^{n-1}(-1)^n \right] (m^2 - 16k - 16) = \frac{(m^2 - 16k - 16)}{k}.$$

Theorem 3.3. For arbitrary integer $n, r \geq 0$, we have following equalities

$$M_{k,n-r+1} = (-1)^r (k)^{1-r} \left[M_{k,n+1}P_{k,r-1} - M_{k,n}P_{k,r} \right],$$

$$M_{k,n-r} = (-1)^r (k)^{1-r} \left[M_{k,n}P_{k,r-1} - M_{k,n-1}P_{k,r} \right],$$

$$M_{k,n-r-1} = (-1)^r (k)^{-r} \left[M_{k,n-1}P_{k,r+1} - M_{k,n}P_{k,r} \right].$$

Proof: It is obvious

$$L^{n-r} = \frac{1}{(-1)^r k^r} \begin{pmatrix} kP_{k,r-1} & -kP_{k,r} \\ -P_{k,r} & P_{k,r+1} \end{pmatrix} L^n$$

and

$$\begin{pmatrix} M_{k,n-r+1} & M_{k,n-r} \\ M_{k,n-r} & M_{k,n-r-1} \end{pmatrix} = L^{n-r} \begin{pmatrix} m+4 & 4 \\ 4 & (m-4)/k \end{pmatrix}. \quad (3.3.1)$$

Otherwise we can write,

$$L^{n-r}\begin{pmatrix} m+4 & 4 \\ 4 & (m-4)/k \end{pmatrix} = \frac{1}{(-1)^r k^r} L^n \begin{pmatrix} kP_{k,r-1} & -kP_{k,r} \\ -P_{k,r} & P_{k,r+1} \end{pmatrix}\begin{pmatrix} m+4 & 4 \\ 4 & (m-4)/k \end{pmatrix}. \qquad (3.3.2)$$

From the (3.3.1) and (3.3.2), we have

$$\begin{pmatrix} M_{k,n-r+1} & M_{k,n-r} \\ M_{k,n-r} & M_{k,n-r-1} \end{pmatrix} = \frac{1}{(-1)^r k^r}\begin{pmatrix} kP_{k,r-1} & -kP_{k,r} \\ -P_{k,r} & P_{k,r+1} \end{pmatrix}\begin{pmatrix} M_{k,n+1} & M_{k,n} \\ M_{k,n} & M_{k,n-1} \end{pmatrix}.$$

Then we have the following results from the above matrix equality

$$M_{k,n-r+1} = (-1)^r (k)^{1-r}\left[M_{k,n+1}P_{k,r-1} - M_{k,n}P_{k,r}\right],$$

$$M_{k,n-r} = (-1)^r (k)^{1-r}\left[M_{k,n}P_{k,r-1} - M_{k,n-1}P_{k,r}\right],$$

$$M_{k,n-r-1} = (-1)^r (k)^{-r}\left[M_{k,n-1}P_{k,r+1} - M_{k,n}P_{k,r}\right].$$

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