Asian Journal of Applied Sciences (ISSN: 2321 – 0893)
Volume 04 – Issue 01, February 2016

Some Periodic Non-linear Difference Models

Kemal Uslu[1,*], Tugba Ugurlu[2]

[1] Selcuk University, Science Faculty, Department of Mathematics
Konya, Turkey

[2] Selcuk University, Science Faculty, Department of Mathematics
Konya, Turkey

[*] *Corresponding author's email: kuslu [AT] selcuk.edu.tr*

ABSTRACT— ***In this study, we analyse the solutions of some periodic non-linear difference equation systems. Then we obtain the period of solutions of these systems.***

Keywords— Periodic discrete system, non-linear discrete system

---

1. INTRODUCTION

Non-linear difference equation systems have been studied in many branches of mathematics as well as other sciences for years. Because of derivatives of many complex behavior based on simple formulation, it is a very interesting subject. There are many applications in science branches and it is easy to understand. This tutorial is to introduce a simple difference equation system, the period of this system, it's solution and it's dynamics [1-5]. One can encounter many investigations and interest in the field of functions of difference equations. Some of them are follows: In [3-4], Cinar C., Yalcinkaya I. and Irıcanin B., Stevic S. took into consideration some systems of non-linear difference equations of higher order with periodic solutions. Nasri M. and at al. introduced a deterministic model for HIV infection in the presence of combination therapy related to difference equations system [2]. Clark and Kulenovic, in [1], investigated the global stability properties and asymptotic behavior of solutions of the recursive equations system. Uslu K. obtained discrete asymptotic stability conditions of perturbed linear discrete systems with periodic coefficients in [5]. Kose H., Uslu K. and Taskara N. examined the dynamics of solutions of non-linear recursive systems in [8]. Then, in [6,7, 8, 9], some non-linear discrete systems have been studied and examinated dynamics of these systems.

In this study, we consider following equation systems

$$x_{n+1}=\frac{1}{(x_{n-1}+y_{n-1})},\quad y_{n+1}=\frac{1}{(z_{n-1}-x_{n-1})}-\frac{1}{(x_{n-1}+y_{n-1})}, \tag{1.1}$$

$$z_{n+1}=\frac{x_{n-1}}{x_n(x_{n-2}+y_{n-2})}+\frac{1}{(x_{n-1}+y_{n-1})}$$

with initial values $x_{-2},x_{-1},x_0,y_{-2},y_{-1},y_0,z_{-1},z_0\,(z_{-1}-x_{-1}\neq 0, z_0-x_0\neq 0)\in\mathbb{R}^{+}$ and

$$x_{n+1}=\frac{1}{(x_{n-2}+y_{n-2})},\quad y_{n+1}=\frac{1}{(z_{n-2}-x_{n-2})}-\frac{1}{(x_{n-2}+y_{n-2})}, \tag{1.2}$$

$$z_{n+1} = \frac{x_{n-1}}{x_n(x_{n-3}+y_{n-3})} + \frac{1}{(x_{n-2}+y_{n-2})}$$

with initial values $x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0, z_{-2}, z_{-1}, z_0\ (z_{-2}-x_{-2} \neq 0, z_{-1}-x_{-1} \neq 0, z_0 - x_0 \neq 0) \in \mathbb{R}^+$.

Firstly, we give basic preliminary definitions:

Let I_1, I_2 and I_3 be some intervals of real numbers and let $F_1 : I_1 \times I_2 \to I_1$, $F_2 : I_1 \times I_2 \times I_3 \to I_2$, $F_3 : I_1 \times I_2 \to I_3$ be three continuously differentiable functions. For every initial condition $(x_i, y_i, z_i) \in I_1 \times I_2 \times I_3$, it is obvious that the system of difference equations

$$x_{n+1} = F_1(x_n, y_n), \qquad y_{n+1} = F_2(x_n, y_n, z_n), \qquad z_{n+1} = F_3(x_n, y_n) \tag{1.3}$$

has a unique solution $\{x_n, y_n, z_n\}_{n=0}^{\infty}$. By using [2], we can write the following results for our this study.

a) A solution $\{x_n, y_n, z_n\}_{n=0}^{\infty}$ of the system of difference equations (1.3) is periodic if there exist a positive integer *p* such that $x_{n+p} = x_n,\ y_{n+p} = y_n,\ z_{n+p} = z_n$, the smallest such positive integer *p* is called the prime period of the solution of difference equation system (1.3).

b) A point $(\bar{x}, \bar{y}, \bar{z}) \in I_1 \times I_2 \times I_3$ is called an equilibrium point of system (1.3) if $\bar{x} = F_1(\bar{x}, \bar{y}),\quad \bar{y} = F_2(\bar{x}, \bar{y}, \bar{z}),\quad \bar{z} = F_3(\bar{x}, \bar{y})$.

c) The equilibrium point $(\bar{x}, \bar{y}, \bar{z})$ of difference equation system (1.3) is called stable (or locally stable) if for every $\varepsilon > 0$, there exist $\delta > 0$, such that for all $(x_s, y_s, z_s) \in I_1 \times I_2 \times I_3$ with $\left\| (x_s, y_s, z_s) - (\bar{x}, \bar{y}, \bar{z}) \right\| < \delta$, implies $\left\| (x_n, y_n, z_n) - (\bar{x}, \bar{y}, \bar{z}) \right\| < \varepsilon$ for all $n \geq 0$. Otherwise equilibrium point is called unstable.

d) The equilibrium point $(\bar{x}, \bar{y}, \bar{z})$ of the difference equation system (1.3) is called asymptotically stable (or locally asymptotically stable), if it is stable and there exist $\gamma > 0$ such that for all $(x_s, y_s, z_s) \in I_1 \times I_2 \times I_3$ with $\left\| (x_s, y_s, z_s) - (\bar{x}, \bar{y}, \bar{z}) \right\| < \gamma$, implies $\lim_{n\to\infty} \left\| (x_n, y_n, z_n) - (\bar{x}, \bar{y}, \bar{z}) \right\| = 0$.

e) The equilibrium point $(\bar{x}, \bar{y}, \bar{z})$ of difference equation system (1.3) is called global asymptotically stable, if it is stable and for every $(x_s, y_s, z_s) \in I_1 \times I_2 \times I_3$, we have $\lim_{n\to\infty} \left\| (x_n, y_n, z_n) - (\bar{x}, \bar{y}, \bar{z}) \right\| = 0$.

2. SOME PERIDIC NON-LINEAR DIFFERENCE MODELS

Theorem 2.1. Suppose that $\{x_n, y_n, z_n\}$ are the solutions of the difference equation system (1.1) with initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0, z_{-1}, z_0\ (z_{-1} - x_{-1} \neq 0, z_0 - x_0 \neq 0) \in \mathbb{R}^+$. Then all solutions of the system (1.1) are periodic with period 6.

Proof: From the system (1.1), it is obtained the following equalities

$$x_{n+1}=\frac{1}{(x_{n-1}+y_{n-1})},\ y_{n+1}=\frac{1}{(z_{n-1}-x_{n-1})}-\frac{1}{(x_{n-1}+y_{n-1})},\ z_{n+1}=\frac{x_{n-1}}{x_n(x_{n-2}+y_{n-2})}+\frac{1}{(x_{n-1}+y_{n-1})}$$

$$x_{n+2}=\frac{1}{(x_n+y_n)},\ \ y_{n+2}=\frac{1}{(z_n-x_n)}-\frac{1}{(x_n+y_n)},\ \ z_{n+2}=x_n+\frac{1}{(x_n+y_n)}$$

$$x_{n+3}=(z_{n-1}-x_{n-1}),\ \ y_{n+3}=\frac{x_n(x_{n-2}+y_{n-2})}{x_{n-1}}-(z_{n-1}-x_{n-1}),\ \ z_{n+3}=\frac{1}{(x_{n-1}+y_{n-1})}+(z_{n-1}-x_{n-1})$$

$$x_{n+4}=(z_n-x_n),\ \ y_{n+4}=\frac{1}{x_n}-(z_n-x_n),\ \ z_{n+4}=\frac{1}{(x_n+y_n)}+(z_n-x_n)$$

$$x_{n+5}=\frac{x_{n-1}}{x_n(x_{n-2}+y_{n-2})},\ \ y_{n+5}=(x_{n-1}+y_{n-1})-\frac{x_{n-1}}{x_n(x_{n-2}+y_{n-2})},\ \ z_{n+5}=(z_{n-1}-x_{n-1})+\frac{x_{n-1}}{x_n(x_{n-2}+y_{n-2})}$$

$$x_{n+6}=x_n,\ \ y_{n+6}=y_n,\ \ z_{n+6}=z_n.$$

Thus all solutions of the system (1.1) are periodic with 6 period.

Theorem 2.2. Suppose that $\{x_n, y_n, z_n\}$ are the solutions of the difference equation system (1.2) with initial values $x_{-3}, x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0, z_{-2}, z_{-1}, z_0\ (z_{-2}-x_{-2}\neq 0, z_{-1}-x_{-1}\neq 0, z_0-x_0\neq 0)\in\mathbb{R}^+$. Then all solutions of the system (1.2) are periodic with period 8.

Proof: From the system (1.2), it is obtained the following equalities

$$x_{n+1}=\frac{1}{(x_{n-2}+y_{n-2})},\ y_{n+1}=\frac{1}{(z_{n-2}-x_{n-2})}-\frac{1}{(x_{n-2}+y_{n-2})},\ z_{n+1}=\frac{x_{n-1}}{x_n(x_{n-3}+y_{n-3})}+\frac{1}{(x_{n-2}+y_{n-2})}$$

$$x_{n+2}=\frac{1}{(x_{n-1}+y_{n-1})},\ y_{n+2}=\frac{1}{(z_{n-1}-x_{n-1})}-\frac{1}{(x_{n-1}+y_{n-1})},\ z_{n+2}=x_n+\frac{1}{(x_{n-1}+y_{n-1})}$$

$$x_{n+3}=\frac{1}{(x_n+y_n)},\ y_{n+3}=\frac{1}{(z_n-x_n)}-\frac{1}{(x_n+y_n)},\ z_{n+3}=\frac{1}{(x_{n-2}+y_{n-2})}+\frac{1}{(x_n+y_n)}$$

$$x_{n+4}=(z_{n-2}-x_{n-2}),\; y_{n+4}=\frac{x_n(x_{n-3}+y_{n-3})}{x_{n-1}}-(z_{n-2}-x_{n-2}),\; z_{n+4}=\frac{1}{(x_{n-1}+y_{n-1})}+(z_{n-2}-x_{n-2})$$

$$x_{n+5}=(z_{n-1}-x_{n-1}),\; y_{n+5}=\frac{1}{x_n}-(z_{n-1}-x_{n-1}),\; z_{n+5}=\frac{1}{(x_n+y_n)}+(z_{n-1}-x_{n-1})$$

$$x_{n+6}=(z_n-x_n),\; y_{n+6}=(x_{n-2}+y_{n-2})-(z_n-x_n),\; z_{n+6}=(z_{n-2}-x_{n-2})+(z_n-x_n)$$

$$x_{n+7}=\frac{x_{n-1}}{x_n(x_{n-3}+y_{n-3})},\; y_{n+7}=(x_{n-1}+y_{n-1})-\frac{x_{n-1}}{x_n(x_{n-3}+y_{n-3})},\; z_{n+7}=(z_{n-1}-x_{n-1})+\frac{x_{n-1}}{x_n(x_{n-3}+y_{n-3})}$$

$$x_{n+8}=x_n,\quad y_{n+8}=y_n,\quad z_{n+8}=z_n$$

Thus all solutions of the system (1.2) are periodic with 8 period.

Theorem 2.3. Suppose that $\{x_n,y_n,z_n\}_{n=0}^{\infty}$ are the solutions of the difference equation system (1.1) with initial values $x_{-2}=a, x_{-1}=b, x_0=c, y_{-2}=d, y_{-1}=e, y_0=f, z_{-1}=g, z_0=h(g\neq a, h\neq c)\in\mathbb{R}^+$. In this case, for $n\geq 0$, all solutions of system (1.1) are

$$x_{6k+1}=\frac{1}{b+e},\; y_{6k+1}=\frac{1}{g-b}-\frac{1}{b+e},\; z_{6k+1}=\frac{b}{c(a+d)}+\frac{1}{b+e}$$

$$x_{6k+2}=\frac{1}{c+f},\quad y_{6k+2}=\frac{1}{h-c}-\frac{1}{c+f},\quad z_{6k+2}=c+\frac{1}{c+f}$$

$$x_{6k+3}=g-b,\quad y_{6k+3}=\frac{c(a+d)}{b}-(g-b),\quad z_{6k+3}=\frac{1}{b+e}+(g-b)$$

$$x_{6k+4}=h-c,\quad y_{6k+4}=\frac{1}{c}-(h-c),\quad z_{6k+4}=\frac{1}{c+f}+(h-c)$$

$$x_{6k+5}=\frac{b}{c(a+d)},\quad y_{6k+5}=(b+e)-\frac{b}{c(a+d)},\quad z_{6k+5}=(g-b)+\frac{b}{c(a+d)}$$

$$x_{6k+6}=c,\quad y_{6k+6}=f,\quad z_{6k+6}=h.$$

Proof: By using induction method, it is obvious that above results hold for $n=0$. Assume that these equalities hold. Now we must show that above results hold for $n=k+1$.

$$x_{6k+7}=\frac{1}{(x_{6k+5}+y_{6k+5})}=\frac{1}{b+e},\qquad y_{6k+7}=\frac{1}{(z_{6k+5}-x_{6k+5})}-\frac{1}{(x_{6k+5}+y_{6k+5})}=\frac{1}{g-b}-\frac{1}{b+e},$$

$$z_{6k+7}=\frac{x_{6k+5}}{x_{6k+6}(x_{6k+4}+y_{6k+4})}+\frac{1}{(x_{6k+5}+y_{6k+5})}=\frac{b}{c(a+d)}+\frac{1}{b+e}$$

$$x_{6k+8}=\frac{1}{(x_{6k+6}+y_{6k+6})}=\frac{1}{c+f},\quad y_{6k+8}=\frac{1}{(z_{6k+6}-x_{6k+6})}-\frac{1}{(x_{6k+6}+y_{6k+6})}=\frac{1}{h-c}-\frac{1}{c+f},$$

$$z_{6k+8}=x_{6k+6}+\frac{1}{(x_{6k+6}+y_{6k+6})}=c+\frac{1}{c+f}$$

$$x_{6k+9}=(z_{6k+5}-x_{6k+5})=g-b,\quad y_{6k+9}=\frac{x_{6k+6}(x_{6k+4}+y_{6k+4})}{x_{6k+5}}-(z_{6k+5}-x_{6k+5})=\frac{c(a+d)}{b}-(g-b),$$

$$z_{6k+9}=\frac{1}{(x_{6k+5}+y_{6k+5})}+(z_{6k+5}-x_{6k+5})=\frac{1}{b+e}+(g-b)$$

$$x_{6k+10}=(z_{6k+6}-x_{6k+6})=h-c,\quad y_{6k+10}=\frac{1}{x_{6k+6}}-(z_{6k+6}-x_{6k+6})=\frac{1}{c}-(h-c),$$

$$z_{6k+10}=\frac{1}{(x_{6k+6}+y_{6k+6})}+(z_{6k+6}-x_{6k+6})=\frac{1}{c+f}+(h-c)$$

$$x_{6k+11}=\frac{x_{6k+5}}{x_{6k+6}(x_{6k+4}+y_{6k+4})}=\frac{b}{c(a+d)},\quad y_{6k+11}=(x_{6k+5}+y_{6k+5})-\frac{x_{6k+5}}{x_{6k+6}(x_{6k+4}+y_{6k+4})}=(b+e)-\frac{b}{c(a+d)},$$

$$z_{6k+11}=(z_{6k+5}-x_{6k+5})+\frac{x_{6k+5}}{x_{6k+6}(x_{6k+4}+y_{6k+4})}=(g-b)+\frac{b}{c(a+d)}$$

$$x_{6k+12}=x_{6k+6}=c,\quad y_{6k+12}=y_{6k+6}=f,\quad z_{6k+12}=z_{6k+6}=h.$$

Theorem 2.4. Suppose that $\{x_n, y_n, z_n\}_{n=0}^{\infty}$ are the solutions of the difference equation system (1.2) with initial values $x_{-3}=a, x_{-2}=b, x_{-1}=c, x_0=d, y_{-3}=e, y_{-2}=f, y_{-1}=g, y_0=h, z_{-2}=i, z_{-1}=l, z_0=m (i\neq b, l\neq c, m\neq d)\in\mathbb{R}^+$. In this case, for $n\geq 0$, all solutions of system (1.2) are

$$x_{8k+1}=\frac{1}{(b+f)},\quad y_{8k+1}=\frac{1}{(i-b)}-\frac{1}{(b+f)},\quad z_{8k+1}=\frac{c}{d(a+e)}+\frac{1}{(b+f)}$$

$$x_{8k+2}=\frac{1}{(c+g)},\quad y_{8k+2}=\frac{1}{(l-c)}-\frac{1}{(c+g)},\quad z_{8k+2}=d+\frac{1}{(c+g)}$$

$$x_{8k+3}=\frac{1}{(d+h)},\quad y_{8k+3}=\frac{1}{(m-d)}-\frac{1}{(d+h)},\quad z_{8k+3}=\frac{1}{(b+f)}+\frac{1}{(d+h)}$$

$$x_{8k+4}=(i-b),\qquad y_{8k+4}=\frac{d(a+e)}{c}-(i-b),\qquad z_{8k+4}=\frac{1}{(c+g)}+(i-b)$$

$$x_{8k+5}=(l-c),\qquad y_{8k+5}=\frac{1}{d}-(l-c),\qquad z_{8k+5}=\frac{1}{(d+h)}+(l-c)$$

$$x_{8k+6}=(m-d),\qquad y_{8k+6}=(b+f)-(m-d),\qquad z_{8k+6}=(i-b)+(m-d)$$

$$x_{8k+7}=\frac{c}{d(a+e)},\qquad y_{8k+7}=(c+g)-\frac{c}{d(a+e)},\qquad z_{8k+7}=(l-c)+\frac{c}{d(a+e)}$$

$$x_{8k+8}=d,\qquad y_{8k+8}=h,\qquad z_{8k+8}=m$$

Proof: By using induction method, it is obvious that above results hold for $n=0$. Assume that these equalities hold. Now we must show that above results hold for $n=k+1$.

$$x_{8k+9}=\frac{1}{(x_{8k+6}+y_{8k+6})}=\frac{1}{(b+f)},\quad y_{8k+9}=\frac{1}{(z_{8k+6}-x_{8k+6})}-\frac{1}{(x_{8k+6}+y_{8k+6})}=\frac{1}{(i-b)}-\frac{1}{(b+f)},$$

$$z_{8k+9}=\frac{x_{8k+7}}{x_{8k+8}(x_{8k+5}+y_{8k+5})}+\frac{1}{(x_{8k+6}+y_{8k+6})}=\frac{c}{d(a+e)}+\frac{1}{(b+f)}$$

$$x_{8k+10}=\frac{1}{(x_{8k+7}+y_{8k+7})}=\frac{1}{(c+g)},\ \ y_{8k+10}=\frac{1}{(z_{8k+7}-x_{8k+7})}-\frac{1}{(x_{8k+7}+y_{8k+7})}=\frac{1}{(l-c)}-\frac{1}{(c+g)},$$

$$z_{8k+10}=x_{8k+8}+\frac{1}{(x_{8k+7}+y_{8k+7})}=d+\frac{1}{(c+g)}$$

$$x_{8k+11}=\frac{1}{(x_{8k+8}+y_{8k+8})}=\frac{1}{(d+h)},\ \ y_{8k+11}=\frac{1}{(z_{8k+8}-x_{8k+8})}-\frac{1}{(x_{8k+8}+y_{8k+8})}=\frac{1}{(m-d)}-\frac{1}{(d+h)},$$

$$z_{8k+11}=\frac{1}{(x_{8k+6}+y_{8k+6})}+\frac{1}{(x_{8k+8}+y_{8k+8})}=\frac{1}{(b+f)}+\frac{1}{(d+h)}$$

$$x_{8k+12}=(z_{8k+6}-x_{8k+6})=(i-b),\ \ y_{8k+12}=\frac{x_{8k+8}(x_{8k+5}+y_{8k+5})}{x_{8k+7}}-(z_{8k+6}-x_{8k+6})=\frac{d(a+e)}{c}-(i-b),$$

$$z_{8k+12}=\frac{1}{(x_{8k+7}+y_{8k+7})}+(z_{8k+6}-x_{8k+6})z_{8k+4}=\frac{1}{(c+g)}+(i-b)$$

$$x_{8k+13}=(z_{8k+7}-x_{8k+7})=(l-c),\qquad y_{8k+13}=\frac{1}{x_{8k+8}}-(z_{8k+7}-x_{8k+7})=\frac{1}{d}-(l-c),$$

$$z_{8k+13}=\frac{1}{(x_{8k+8}+y_{8k+8})}+(z_{8k+7}-x_{8k+7})=\frac{1}{(d+h)}+(l-c)$$

$$x_{8k+14}=(z_{8k+8}-x_{8k+8})=(m-d),\qquad y_{8k+14}=(x_{8k+6}+y_{8k+6})-(z_{8k+8}-x_{8k+8})=(b+f)-(m-d),$$

$$z_{8k+14}=(z_{8k+6}-x_{8k+6})+(z_{8k+8}-x_{8k+8})z_{8k+6}=(i-b)+(m-d)$$

$$x_{8k+15}=\frac{x_{8k+7}}{x_{8k+8}(x_{8k+5}+y_{8k+5})}=\frac{c}{d(a+e)},\quad y_{8k+15}=(x_{8k+7}+y_{8k+7})-\frac{x_{8k+7}}{x_{8k+8}(x_{8k+5}+y_{8k+5})}=(c+g)-\frac{c}{d(a+e)},$$

$$z_{8k+15}=(z_{8k+7}-x_{8k+7})+\frac{x_{8k+7}}{x_{8k+8}(x_{8k+5}+y_{8k+5})}=(l-c)+\frac{c}{d(a+e)}$$

$$x_{8k+16}=d,\qquad y_{8k+16}=h,\qquad z_{8k+16}=m.$$

3. REFERENCES

[1] Clark D., Kulenovic M.R.S., "A coupled system of rational difference equations", Computers and Mathematics with Applications, 43, 849-867, 2002.

[2] Nasri M., Dehghan M. and Douraki M.J., Mathias R., "Study of a system of non-linear difference equations arising in a deterministic model for HIV infection", Applied Mathematics and Computation, 171, 1306-1330, 2005.

[3] Cinar C., Yalcinkaya I., "On the positive solutions of difference equation system, International Mathematical Journal, vol. 5, 2004.

[4] Iricanin B., Stevic S., "Some systems of non-linear difference equations of higher order with periodic solutions", Dynamic of Continuous, Discrete and Impulse Sysetems, Series A Mathematical Analysis, vol.13, 499-507, 2006.

[5] Uslu K., "On the discrete asymptotic stability conditions of perturbed linear discrete systems with periodic coefficients", Anziam Journal, 48(1), 135-142, 2006.

[6] Taskara N., Uslu K., Tollu D. T., "The periodicity and solutions of the ratinal difference equation with periodic coefficients", Computers & Mathematics with Applications, 62(4), 1807-1813, 2011.

[7] Hanedar E.,Uslu K, "The stability conditions related to equilibrium points of a difference systems and the period of solutions of this system", Far East Journal of Applied Mathematics, 85(1-2), 69-74, 2011.

[8] Kose H., Uslu K. Taskara N., "On the dynamics of solutions of non-linear recursive system", Communications in Mathematics and Applications, 2(5), 91-98, 2010.

[9] Uslu K., Taskara N., Hekimoglu O., "On the periodicity and stability conditions of a non-linear system", The First International Conference on Mathematics and Statistics, American University of Sharjah, UAE, 110 pp., 2010.