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On Fuzzy and Anti Fuzzy Γ-Ideals in Γ-CI-Algebras

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ABSTRACT— ***The purpose of this paper is to introduce the notion of a fuzzy Γ-ideals in Γ-CI-algebras, we study Γ-ideals, upper sets, fuzzy and anti fuzzy Γ-ideals in Γ-CI-algebras. Those are extended from upper sets, fuzzy and anti fuzzy ideals in CI-algebras respectively. Some characterizations of Γ-ideals, upper sets, fuzzy and anti fuzzy Γ-ideals in Γ-CI-algebras are obtained. Moreover, we investigate the relationships between fuzzy and anti fuzzy Γ-ideals in Γ-CI-algebras.***

Keywords— Γ-CI-algebra, Γ-ideal, upper set, fuzzy Γ-ideal, anti fuzzy Γ-ideal.

1. INTRODUCTION

In what follows, let X denote an CI-algebra unless otherwise specified. By an CI-algebra we mean an algebra $(X;*,1)$ of type (2, 0) with a single binary operation "$*$" that satisfies the following identities: for any $x, y, z \in X$,

1. $x*x=1$,
2. $1*x=x$,
3. $x*(y*z)=y*(x*z)$.

In, 2006, H. S. Kim and Y. H. Kim defined a BE-algebra. Biao Long Meng, defined notion of CI–algebra as a generation of a BE–algebra [9]. BE-algebras and CI-algebras are studied in detail be some researchers [3, 7, 9, 11] and some fundamental properties of CI–algebra are discussed.

The concept of an ordered Γ-CI-algebras was first given by Pairote Yiarayong and Phakakorn Panpho in [12] which is infect the generalization of a CI-algebras.

In this paper is to introduce the notion of a Γ-CI-algebras, we study Γ-ideals, upper sets, fuzzy and anti fuzzy Γ-ideals in Γ-CI-algebras. Those are extended from upper sets, fuzzy and anti fuzzy ideals in CI-algebras respectively. Some characterizations of Γ-ideals, upper sets, fuzzy and anti fuzzy Γ-ideals in Γ-CI-algebras are obtained. Moreover, we investigate the relationships between fuzzy and anti fuzzy Γ-ideals in Γ-CI-algebras.

2. BASIC PROPERTIES

In this section we refer to [12, 13] for some elementary aspects and quote few definitions and examples which are essential to step up this study. For more detail we refer to the papers in the references.

Definition 2.1. [12, 13] Let X and Γ be any nonempty sets. If there exists a mapping $X\times\Gamma\times X\to X$ written as (x,γ,y) by $x\gamma y$, then $(\Gamma,X;1)$ is called a Γ-CI-algebra if

1. $x\gamma x = 1$ for all $x\in X$ and $\gamma\in\Gamma$,
2. $1\gamma x = x$ for all $x\in X$ and $\gamma\in\Gamma$,
3. $x\gamma(y\beta z) = y\gamma(x\beta z)$

for all $x,y,z\in X$ and $\gamma,\beta\in\Gamma$.

Lemma 2.2. [12] Let X be a Γ-CI-algebra. Then $x\gamma y = x\beta y$ for any $x, y \in X$ and $\gamma, \beta \in \Gamma$.

Proposition 2.3. [12] Any Γ-CI-algebra X satisfies for any $x, y \in X$ and $\gamma, \beta, \alpha \in \Gamma$, $y\gamma\left[\left(y\alpha x\right)\beta x\right] = 1$

Proposition 2.4. [12] Any Γ-CI-algebra X satisfies for any $x, y \in X$ and $\gamma, \beta \in \Gamma$, $\left(x\gamma 1\right)\beta\left(y\gamma 1\right) = \left(x\gamma y\right)\beta 1$.

Proposition 2.5. [12] Let X be a Γ-CI-algebra. If $x\gamma(x\beta y) = x\beta y$ for any $x, y \in X$ and $\gamma, \beta \in \Gamma$, then $x\gamma 1 = 1$.

Definition 2.6. [13] Let X be a Γ-CI-algebra and A a nonempty subset of X. A is said to be a Γ-subalgebra of X if $AA \subseteq A$.

Definition 2.7. [13] Let X be a Γ-CI-algebra and A a nonempty subset of X. A is said to be a Γ-ideal of X if it satisfies: $x \in X$ and $\gamma, \beta, \alpha \in \Gamma$

1. $X\Gamma A \subseteq A$,
2. $a, b \in A$ imply that $(a\gamma(b\beta x))\alpha x \in A$.

For any Γ-CI-algebra X, $\{1\}$ and X are trivial ideals (resp. Γ-subalgebras) of X. Obviously every ideal in a Γ-CI-algebra is a Γ-subalgebra.

Lemma 2.8. [13] Let X be a Γ-CI-algebra. Then

(1) Every Γ-ideal of X contains 1,

(2) If A is a Γ-ideal of X, then $(a\gamma x)\beta x \in A$ for all $a \in A, x \in X$ and $\gamma, \beta \in A$.

Definition 2.9.[13] A Γ-CI-algebra X is said to be Γ-transitive if for all $x, y, z \in X$ and $\gamma, \alpha, \beta, \delta, \lambda \in \Gamma$, $(y\gamma z)\alpha((x\beta y)\delta(x\lambda z)) = 1$.

Definition 2.10. [13] A Γ-CI-algebra $\left(\Gamma, X; 1\right)$ is said to be Γ-self-distributive if $x\gamma(y\beta z) = (x\gamma y)\beta(x\gamma z)$ for all $x, y, z \in X$ and $\gamma, \beta \in \Gamma$.

Definition 2.12. [13] Let X be a Γ-CI-algebra and $a, b \in X, \gamma, \beta \in \Gamma$. Define $A\left(a\gamma, b\beta\right)$ by

$$A\left(a\gamma, b\beta\right) = \{1\} \cup \left\{x \in X : a\gamma\left(b\beta x\right) = 1\right\}.$$

We call $A\left(a\gamma, b\beta\right)$ an upper set of a and b.

Remark.

1. By Definition 2.12, we have $A\left(a\gamma, b\beta\right) = A\left(b\gamma, a\beta\right)$ for all $a, b \in X$ and $\gamma, \beta \in \Gamma$.
2. By Definition 2.12, we have

$$\begin{aligned} a\gamma\left(b\alpha a\right) &= b\gamma\left(a\alpha a\right) \\ &= b\gamma 1 \\ &= b\gamma\left(1\alpha 1\right) \\ &= \left(b\gamma 1\right)\alpha\left(b\gamma 1\right) \end{aligned}$$

$= \quad 1.$

Then $a \in A(a\gamma, b\alpha)$.

3. FUZZY and ANTI FUZZY Γ-IDEALS IN Γ-CI-ALGEBRAS

The results of the following lemmas seem to play an important role to study Γ-CI-algebra; these facts will be used frequently and normally we shall make no reference to this definitions.

Definition 3.1. Let X be a non-empty set**.** A fuzzy subset μ of the set X is a mapping $\mu : X \to [0,1]$. The complement of μ denoted by μ^c, is the fuzzy set in X given by $\mu^c(x) = 1 - \mu(x)$, for all $x \in X$.

Definition 3.2. Let X be a Γ-CI-algebra. A fuzzy set μ in X is called a fuzzy Γ-ideal of X if

1. $\mu(x\gamma y) \geq \mu(y)$, for all $x, y \in X$ and $\gamma \in \Gamma$.
2. $\mu((x\gamma(y\,\alpha z))\beta z) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y, z \in X$ and $\gamma, \alpha, \beta \in \Gamma$.

Lemma 3.3. Let X be a Γ-CI-algebra and let $x \in X$. If μ is a fuzzy Γ-ideal of X, then $\mu(1) \geq \mu(x)$.

Proof. By Definition 3.2, we get $\mu(1) = \mu(x\gamma x) \geq \mu(x)$ for all $x \in X, \gamma \in \Gamma$. Then $\mu(1) \geq \mu(x)$.

Theorem 3.4. Let X be a Γ-CI-algebra and let $x, y \in X, \gamma, \alpha \in \Gamma$. If μ is a fuzzy Γ-ideal of X, then $\mu((x\gamma y)\alpha y) \geq \mu(x)$

Proof. By Lemma 3.3, we get

$$\begin{aligned} \mu((x\gamma y)\alpha y) &= \mu((x\gamma(1\beta y))\alpha y) \\ &\geq \min\{\mu(x), \mu(1)\} \\ &= \mu(x). \end{aligned}$$

for all $x, y \in X, \gamma, \beta, \alpha \in \Gamma$.

Theorem 3.5. Let X be a Γ-CI-algebra and let $x \leq y$ for all $x, y \in X$. If μ is a fuzzy Γ-ideal of X, then $\mu(y) \leq \mu(x)$.

Proof. Let μ be a fuzzy Γ-ideal of a Γ-CI- algebra X and let $x, y \in X, \gamma \in \Gamma$ such that $x \leq y$ then $x\gamma y = 1$. Now

$$\begin{aligned} \mu(y) &= \mu(1\alpha y) \\ &= \mu((x\gamma y)\alpha y) \\ &\leq \mu(x). \end{aligned}$$

Hence $\mu(y) \leq \mu(x)$.

Theorem 3.6. Let X be a Γ-CI-algebra and let $x \le y, \mu(x\gamma z) \ge \min\{\mu(x\alpha(y\beta z)), \mu(y)\}$, for all $x, y, z \in X, \gamma, \alpha, \beta \in \Gamma$. If μ is a fuzzy Γ-ideal of X, then $\mu(x) \le \mu(y)$.

Proof. Let μ be a fuzzy Γ-ideal of a Γ-CI- algebra X and let $x, y \in X, \gamma, \alpha, \beta \in \Gamma$ such that $x \le y$ then $x\beta y = 1$. Now

$$\begin{aligned} \mu(y) &= \mu(1\gamma y) \\ &\ge \min\{\mu(1\alpha(x\beta y)),\ \mu(x)\} \\ &= \min\{\mu(1\alpha 1), \mu(x)\} \\ &= \min\{\mu(1), \mu(x)\} \\ &= \mu(x). \end{aligned}$$

Hence $\mu(x) \le \mu(y)$.

Theorem 3.7. Let X be a Γ-transitive Γ-CI-algebra. If μ is a fuzzy Γ-ideal of X, then

$$\mu(x\gamma z) \ge \min\{\mu(x\beta(y\gamma z)), \mu(y)\},$$

for all $x, y, z \in X, \gamma, \alpha, \beta \in \Gamma$.

Proof. Let μ be a fuzzy Γ-ideal of a Γ-transitive Γ-CI- algebra X and let $x, y, z \in X, \gamma, \alpha, \beta \in \Gamma$. Since X is Γ-transitive, we have $((y\gamma z)\alpha z)\beta((x\beta(y\gamma z))\alpha(x\beta z)) = 1$. Then

$$\begin{aligned} \mu(x\gamma z) &= \mu(1(x\gamma z)) \\ &= \mu(((y\gamma z)\alpha z)\beta((x\beta(y\gamma z))\alpha(x\beta z))(x\gamma z)) \\ &\ge \min\{\mu((y\gamma z)\alpha z), \mu((x\beta(y\gamma z))\} \\ &\ge \min\{\mu(y), \mu((x\beta(y\gamma z))\} \end{aligned}$$

Hence $\mu(x\gamma z) \ge \min\{\mu(x\beta(y\gamma z)), \mu(y)\}$.

Theorem 3.8. Let X be a Γ- self-distributive Γ-CI-algebra. If μ is a fuzzy Γ-ideal of X, then

$$\mu(x\gamma z) \ge \min\{\mu(x\beta(y\gamma z)), \mu(y)\},$$

for all $x, y, z \in X, \gamma, \alpha, \beta \in \Gamma$.

Proof. Let μ be a fuzzy Γ-ideal of a Γ-transitive Γ-CI- algebra X and let $x, y, z \in X, \gamma, \alpha, \beta, \delta, \lambda \in \Gamma$. Since

$$\begin{aligned} (y\gamma z)\lambda((x\alpha y)\delta(x\beta z)) &= ((y\gamma z)\lambda(x\alpha y))\delta((y\gamma z)\lambda(x\beta z)) \\ &= ((x\lambda((y\gamma z)\alpha y))\delta(x\lambda((y\lambda z)\beta z))) \\ &= x\lambda(((y\gamma z)\alpha y)\delta((y\gamma z)\beta z)) \\ &= x\lambda((y\gamma z)\delta(y\alpha z)) \\ &= x\lambda 1 \\ &= x\lambda(x\lambda x) \\ &= (x\lambda x)\lambda(x\lambda x) \\ &= 1, \end{aligned}$$

we have X is a Γ-transitive Γ-CI-algebra. By Theorem 3.7, we get $\mu(x\gamma z) \geq \min\{\mu(x\beta(y\gamma z)), \mu(y)\}$.

Definition 3.9. Let X be a Γ-CI-algebra. A fuzzy set μ in X is called an anti fuzzy Γ-ideal of X if

1. $\mu(x\gamma y) \leq \mu(y)$, for all $x, y \in X$ and $\gamma \in \Gamma$.
2. $\mu((x\gamma(y\, \alpha z))\beta z) \leq \max\{\mu(x), \mu(y)\}$, for all $x, y, z \in X$ and $\gamma, \alpha, \beta \in \Gamma$.

Lemma 3.10. Let X be a Γ-CI-algebra and let $x \in X$. If μ is a ant fuzzy Γ-ideal of X, then $\mu(1) \leq \mu(x)$.

Proof. By Definition 3.9, we get $\mu(1) = \mu(x\gamma x) \leq \mu(x)$, for all $x \in X, \gamma \in \Gamma$. Then $\mu(1) \leq \mu(x)$.

Theorem 3.11. Let X be a Γ-CI-algebra and let $x, y \in X, \gamma, \alpha \in \Gamma$. If μ is an anti fuzzy Γ-ideal of X, then $\mu((x\gamma y)\alpha y) \geq \mu(x)$

Proof. By Lemma 3.10, we get

$$\begin{aligned} \mu((x\gamma y)\alpha y) &= \mu((x\gamma(1\beta y))\alpha y) \\ &\leq \max\{\mu(x), \mu(1)\} \\ &= \mu(x), \end{aligned}$$

for all $x, y \in X, \gamma, \beta, \alpha \in \Gamma$.

Theorem 3.12. Let X be a Γ-CI-algebra and let $x \leq y$ for all $x, y \in X$. If μ is an anti fuzzy Γ-ideal of X, then $\mu(x) \leq \mu(y)$.

Proof. Let μ be a anti fuzzy Γ-ideal of a Γ-CI- algebra X and let $x, y \in X, \gamma \in \Gamma$ such that $x \leq y$ then $x\gamma y = 1$. Now

$$\begin{aligned} \mu(y) &= \mu(1\alpha y) \\ &= \mu((x\gamma y)\alpha y) \\ &\geq \mu(x). \end{aligned}$$

Hence $\mu(x) \leq \mu(y)$.

Theorem 3.13. Let X be a Γ-CI-algebra and let $x \leq y, \mu(x\gamma z) \leq \max\{\mu(x\alpha(y\beta z)), \mu(y)\}$, for all $x, y, z \in X, \gamma, \alpha, \beta \in \Gamma$. If μ is an anti fuzzy Γ-ideal of X, then $\mu(x) \geq \mu(y)$.

Proof. Let μ be an anti fuzzy Γ-ideal of a Γ-CI- algebra X and let $x, y \in X, \gamma, \alpha, \beta \in \Gamma$ such that $x \leq y$ then $x\beta y = 1$. Now

$$\begin{aligned} \mu(y) &= \mu(1\gamma y) \\ &\leq \max\{\mu(1\alpha(x\beta y)), \mu(x)\} \\ &= \max\{\mu(1\alpha 1), \mu(x)\} \\ &= \max\{\mu(1), \mu(x)\} \\ &= \mu(x). \end{aligned}$$

Hence $\mu(y) \leq \mu(x)$.

Theorem 3.14. Let X be a Γ-transitive Γ-CI-algebra. If μ is an anti fuzzy Γ-ideal of X, then

$$\mu(x\gamma z) \le \max\{\mu(x\beta(y\gamma z)), \mu(y)\},$$

for all $x, y, z \in X, \gamma, \alpha, \beta \in \Gamma$.

Proof. Let μ be an anti fuzzy Γ-ideal of a Γ-transitive Γ-CI- algebra X and let $x, y, z \in X, \gamma, \alpha, \beta \in \Gamma$. Since X is Γ-transitive, we have $((y\gamma z)\alpha z)\beta((x\beta(y\gamma z))\alpha(x\beta z)) = 1$. Then

$$\begin{aligned} \mu(x\gamma z) &= \mu(1(x\gamma z)) \\ &= \mu(((y\gamma z)\alpha z)\beta((x\beta(y\gamma z))\alpha(x\beta z))(x\gamma z)) \\ &\le \max\{\mu((y\gamma z)\alpha z), \mu((x\beta(y\gamma z))\} \\ &\le \max\{\mu(y), \mu((x\beta(y\gamma z))\} \end{aligned}$$

Hence $\mu(x\gamma z) \le \max\{\mu(x\beta(y\gamma z)), \mu(y)\}$.

Theorem 3.14. Let X be a Γ- self-distributive Γ-CI-algebra. If μ is an anti fuzzy Γ-ideal of X, then

$$\mu(x\gamma z) \le \max\{\mu(x\beta(y\gamma z)), \mu(y)\},$$

for all $x, y, z \in X, \gamma, \alpha, \beta \in \Gamma$.

Proof. Let μ be an anti fuzzy Γ-ideal of a Γ-transitive Γ-CI- algebra X and let $x, y, z \in X, \gamma, \alpha, \beta, \delta, \lambda \in \Gamma$. Since

$$\begin{aligned} (y\gamma z)\lambda((x\alpha y)\delta(x\beta z)) &= ((y\gamma z)\lambda(x\alpha y))\delta((y\gamma z)\lambda(x\beta z)) \\ &= ((x\lambda((y\gamma z)\alpha y))\delta(x\lambda((y\lambda z)\beta z))) \\ &= x\lambda(((y\gamma z)\alpha y)\delta((y\gamma z)\beta z)) \\ &= x\lambda((y\gamma z)\delta(y\alpha z)) \\ &= x\lambda 1 \\ &= x\lambda(x\lambda x) \\ &= (x\lambda x)\lambda(x\lambda x) \\ &= 1, \end{aligned}$$

we have X is a Γ-transitive Γ-CI-algebra. By Theorem 3.14, we get $\mu(x\gamma z) \le \max\{\mu(x\beta(y\gamma z)), \mu(y)\}$.

Theorem 3.15. Let X be a Γ-CI-algebra. Then μ is a fuzzy Γ-ideal of X if and only if μ^c is an anti fuzzy Γ-ideal of X.

Proof. Let μ is a fuzzy Γ-ideal of a Γ-CI-algebra X and let $x, y, z \in X, \gamma, \alpha \in \Gamma$. Then

$$\begin{aligned} \mu^c(x\gamma y) &= 1 - \mu(x\gamma y) \\ &\le 1 - \mu(y) \\ &\le \mu^c(y) \end{aligned}$$

so that $\mu^c(x\gamma y) \le \mu^c(y)$. Since $\mu((x\gamma(y\ \alpha z))\beta z) \le \max\{\mu(x),\ \mu(y)\}$, we get

$$
\begin{aligned}
\mu^c\left(\left(x\gamma\left(y\ \alpha z\right)\right)\beta z\right) &= 1-\mu\left(\left(x\gamma\left(y\ \alpha z\right)\right)\beta z\right)\\
&\leq 1-\min\left\{\mu(x),\ \mu(y)\right\}\\
&= 1-\min\left\{1-\mu^c(x),\ 1\text{-}\mu^c(y)\right\}\\
&= \max\left\{\mu(x),\ \mu(y)\right\}.
\end{aligned}
$$

Thus, μ^c is an anti fuzzy Γ-ideal of X. The converse also can be proved similarly.

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