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On the Classical Prime Radical Formula and Classical Prime of Semimodules

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ABSTRACT— *Let R be a commutative semiring and M an R semimodule. A proper subsemimodule N of M is called a classical prime subsemimodule, if for any $a,b \in R$ and $m \in M, abm \in N$ implies that $am \in N$ or $bm \in N$. We will introduce and study the notion of prime bases for classical prime subsemimodules and utilize them to derive some formulas on the classical prime radical of subsemimodules of a semimodule. In particular, we study some basic properties of prime radical and classical prime radical of subsemimodule in M. Moreover, we investigate relationships between classical prime radical and prime radical of subsemimodule in M.*

Keywords— prime subsemimodule, classical prime subsemimodule, prime radical, classical prime radical, prime ideal

1. INTRODUCTION

Throughout this paper a semiring will be defined as follows: A semiring is a set R together with two binary operations called addition "$+$" and multiplication "$\cdot$" such that $(R,+)$ is a commutative semigroup and $(R,\cdot)$ is semigroup; connecting the two algebraic structures are the distributive laws : $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$ for all $a, b, c \in R.$ A subset A of a semiring R is called an ideal of R if for $a,b \in A$ and $r \in R, a+b \in A, ar \in A$ and $ra \in A.$ A proper ideal P of R is called a prime ideal if $ab \in P,$ where $a,b \in R,$ implies that either $a \in P$ or $b \in P.$ A proper ideal P of R is said to be quasi-primary if for all $a,b \in R, ab \in P$ implies that either $a^n \in P$ or $b^n \in P,$ for some positive integer $n.$ Clearly every prime is a quasi-primary. A semimodule M over a semiring R is a commutative monoid M, together with a function $R \times M \to M,$ defined by $(r,m) \mapsto rm$ such that:

1. $r(m+n) = rm+rn$
2. $(r+s)m = rm+sm$
3. $(rs)m = r(sm)$
4. $1m = m$

for all $m,n \in M$ and $r,s \in R.$ Clearly every ring is a semiring and hence every module over a ring R is a left semimodule over a semiring $R.$ A nonempty subset N of a R-semimodule M is called subsemimodule of M if N is closed under addition and closed under scalar multiplication. A proper subsemimodule N of an R-semimodule M is said to be prime if $rm \in N, r \in R, m \in M,$ then either $m \in N$ or $rM \subseteq N.$

J. Saffar Ardabili, S. Motmaen and A. Yousefian Darani in (2011) defined a different class of subsemimodules and called it classical prime. A proper subsemimodule N of M is said to be classical prime when for $a,b \in R$ and $m \in M, abm \in N$ implies that $am \in N$ or $bm \in N.$ A classical prime radical of N in M, denoted by

$c.rad_M(N)$, is defined to be the intersection of all classical prime subsemimodules containing N. Should there be no classical prime subsemimodule of M containing N, then we put $c.rad_M(N) = M$. In this paper we introduce the concept of classical prime subsemimodules of a semimodule M, and study some basic properties of this class of subsemimodules. Moreover, we investigate relationships between classical prime and prime radical of subsemimodules in M.

2. BASIC PROPERTIES OF SUBSEMIMODULES

In this section we refer to [1, 5] for some elementary aspects and quote few theorem and lemmas which are essential to step up this study. For more details we refer to the papers in the references.

Definition 2.1. [5] Let M be an R-semimodule and N be a proper subsemimodule of M. An associated ideal of N is defined as $(N:M) = \{a \in R:\ aM \subseteq N\}$.

Lemma 2.2. [5] Let M be an R-semimodule and N be a proper subsemimodule of M. If N is a subtractive subsemimodule of M, and let $m \in M$. Then the following hold:

1. $(N:M)$ is a subtractive ideal of R.
2. $(0:M)$ and $(N:m)$ are subtractive ideals of R.

Lemma 2.3. [1] Let N and K be subsemimodules of a semimodule M over a semiring R with $N \subseteq K$. Then K/N is a subsemimodule of M/N.

Definition 2.4. Let N be any subsemimodule of an R-semimodule M. For $a \in R$ and an ideal I of R, the sets $[N:a]$ and $[N:I]$ are defined by

1. $[N:a] = \{m \in M : am \in N\}$ and
2. $[N:I] = \{m \in M : Im \subseteq N\}$.

Remark. Let N be any subsemimodule of an R-semimodule M and let I be an ideal of $R, a \in R$. Then

1. $[N:a] \neq \varnothing$ and $[N:I] \neq \varnothing$
2. $N \subseteq [N:a]$ and $N \subseteq [N:I]$.

Proposition 2.5. Let N be a proper subsemimodule of semimodule M over a semiring R. If N is classical prime subsemimodule of M, then $[N:c]$ is a classical prime subsemimodule of M, where $c \in R$.

Proof. Let $abm \in [N:c]$, where $a,b \in R$ and $m \in M$. By Definition 2.4, we have $(ca)bm = c(abm) \in N$. Since $(ca)bm \in N$ and N is classical prime subsemimodule of M, we have $cam \in N$ or $bm \in N$. Then $am \in [N:c]$ or $bm \in N \subseteq [N:c]$. Hence $[N:c]$ is a classical prime subsemimodule of M.

Proposition 2.6. Let R be a semiring, with identity, and let N be a proper subsemimodule of semimodule M over R. If $[N:c]$ is a classical prime subsemimodule of M, then N is classical prime subsemimodule of M, where $c \in R$.

Proof. Let $abm \in N$, where $a,b \in R$ and $m \in M$. Then $1bm = bm \in [N:a]$. Since $1bm \in [N:a]$ and $[N:a]$ is weakly classical prime subsemimodule M, we have $m = 1m \in N \subseteq [N:a]$ or $bm \in N$. Thus $am \in N$ or $bm \in N$, and hence N is classical prime of M.

3. BASIC PROPERTIES OF (N,P)

The results of the following lemmas seem to be at the heart of the theory of classical prime subsemimodules; these facts will be used so frequently that normally we shall make no reference to this lemma.

Lemma 3.1. If N is a proper subsemimodule of an R-semimodule M and P is a proper ideal of R, then $PM + N \subseteq (N,P) = \{x \in M \mid cx \in PM + N, \text{ where } c \in R - \sqrt{P}\}$.

Proof. Let $x \in PM + N$. Then $x = pm + k$, where $p \in P, m \in M$ and $k \in N$. There exists $1 \in R - \sqrt{P}$ so that $1x = x = pm + k \in PM + N$. It follows that $x \in (N,P)$ and hence $PM + N \subseteq (N,P)$.

Lemma 3.2. If N is a proper subsemimodule of an R-semimodule M and P is a proper ideal of R, then (N,P) is a subsemimodule of M.

Proof. It follows from Lemma 3.1 that $(N,P) \neq \varnothing$. To show that subsemimodule properties of (N,P) hold, let $r \in R$ and $x,y \in (N,P)$. Since P is a proper ideal of R, we have $P \neq R$ so that $R - P \neq \varnothing$. There exists c_1, c_2 with $c_1, c_2 \in R - P$ such that $c_1x, c_2y \in PM + N$. Then $c_1x = p_1m_1 + n_1$ and $c_2y = p_2m_2 + n_2$, where $p_1, p_2 \in P, m_1, m_2 \in M$ and $n_1, n_2 \in N$. Now consider

$$\begin{aligned} c_1c_2(x+y) &= c_1c_2x + c_1c_2y \\ &= c_2(c_1x) + c_1(c_2y) \\ &= c_2(p_1m_1 + n_1) + c_1(p_2m_2 + n_2) \\ &= c_2p_1m_1 + c_2n_1 + c_1p_2m_2 + c_1n_2 \\ &= (c_2p_1m_1 + c_1p_2m_2) + (c_2n_1 + c_1n_2) \in PM + N \end{aligned}$$

and

$$\begin{aligned} c_1rx &= r(c_1x) \\ &= r(p_1m_1 + n_1) \\ &= rp_1m_1 + rn_1 \in PM + N. \end{aligned}$$

Therefore $x + y \in (N,P)$ and $rx \in (N,P)$. Hence (N,P) is a subsemimodule of M.

Proposition 3.3. If N is a proper subsemimodule of an R-semimodule M and P is a prime ideal of R, then $(N,P)=M$ or (N,P) is a prime subsemimodule of M.

Proof. Suppose that $(N,P)\neq M$. We will show that (N,P) is a prime subsemimodule of M. By Lemma 3.2, we have (N,P) is a subsemimodule of M. To see the prime property of (N,P), let $r\in R$ and $m\in M$ such that $rm\in(N,P)$. We will prove that

$$m\in(N,P) \text{ or } rM\subseteq(N,P).$$

Since $rm\in(N,P)$, there exist c with $c\in R-P$, such that $crm\in PM+N$. We have 2 cases to consider; $r\in P$ and $r\notin P$.

Case 1. If $r\in P$, then by Lemma 2.1, we have $rM\subseteq PM\subseteq PM+N\subseteq(N,P)$. Therefore $rM\subseteq(N,P)$.

Case 2. If $r\notin P$, then $cr\notin P$. Thus $m\in(N,P)$. Therefore (N,P) is a prime subsemimodule of M.

Corollary 3.4 If N is a proper subsemimodule of an R-semimodule M and P is a prime ideal of R, then $(N,P)=M$ or (N,P) is a classical prime subsemimodule of M.

Proof. It follows from Lemma 3.3.

3. THE CLASSICAL PRIME RADICAL SUBSEMIMODULES

Before theorizing of the theorem of the relationship between radical and classical prime radical of subsemimodules of an R-semimodule M. Our starting point is the following lemma:

Lemma 4.1. If N and K are subsemimodules of an R-semimodule M such that $N\subseteq K$, then $c.rad_M(N)\subseteq c.rad_M(K)$.

Proof. If there is no classical prime subsemimodule of M containing K, then $c.rad_M(K)=M$. Since $c.rad_M(N)$ is a subsemimodule of M, we have $c.rad_M(N)\subseteq M=c.rad_M(K)$. If there exists a classical prime subsemimodules of M containing K, then $c.rad_M(K)$ is a subsemimodule of M, with $K\subseteq c.rad_M(K)$. Since $N\subseteq K$, we have $N\subseteq c.rad_M(K)$. Therefore $c.rad_M(N)\subseteq c.rad_M(K)$.

Corollary 4.2. If N is a subsemimodule of an R-semimodule M, then $c.rad_M(N)\subseteq c.rad_M(M)=M$.

Proof. It follows from Lemma 3.1.

Proposition 4.3. Let N and K be subsemimodules of an R-semimodule M. Then

(1) $N\subseteq c.rad_M(N)$

(2) $c.rad_M(c.rad_M(N))=c.rad_M(N)$

(3) $c.rad_M(N\cap K)\subseteq c.rad_M(N)\cap c.rad_M(K)$

(4) $c.rad_M(N+K)=c.rad_M(N)+c.rad_M(K)$.

Proof. (1) Obviously, $N\subseteq P$ for every classical prime subsemimodule P. Then by the definition of $c.rad_M(N)$, we have $N\subseteq c.rad_M(N)$.

(2) Since $N \subseteq c.rad_M(N)$, by Lemma 4.1, $c.rad_M(N) \subseteq c.rad_M(c.rad_M(N))$. We will show that $c.rad_M(c.rad_M(N)) \subseteq c.rad_M(N)$. If there is no classical prime subsemimodule of M containing N, then $c.rad_M(N) = M$. Thus $c.rad_M(c.rad_M(N)) \subseteq c.rad_M(N)$. If there exists a classical prime subsemimodule of M containing N, then let W be a classical prime subsemimodule of M containing N. Then by the definition of $c.rad_M(N)$, $c.rad_M(N) \subseteq W$. It follows that $c.rad_M(c.rad_M(N)) \subseteq c.rad_M(N)$ and hence $c.rad_M(c.rad_M(N)) = c.rad_M(N)$.

(3) Since $N \cap K \subseteq N$ and $N \cap K \subseteq K$, by Lemma 4.1, we have $c.rad_M(N \cap K) \subseteq c.rad_M(N)$ and

$$c.rad_M(N \cap K) \subseteq c.rad_M(K).$$

Therefore $c.rad_M(N \cap K) \subseteq c.rad_M(N) \cap c.rad_M(K)$.

(4) We will show that $c.rad_M(N + K) \subseteq c.rad_M(N) + c.rad_M(K)$. It is clear that

$$N + K \subseteq c.rad_M(N) + c.rad_M(K),$$

so that $c.rad_M(N + K) \subseteq c.rad_M(N) + c.rad_M(K)$. On the other hand, we will show that

$$c.rad_M(N) + c.rad_M(K) \subseteq c.rad_M(N + K).$$

Since $N \subseteq N + K$ and $K \subseteq N + K$, we have $c.rad_M(N) \subseteq c.rad_M(N + K)$ and

$$c.rad_M(K) \subseteq c.rad_M(N + K).$$

Thus

$$c.rad_M(N) + c.rad_M(K) \subseteq c.rad_M(N + K).$$

Hence $c.prad_M(N + K) = c.prad_M(N) + c.prad_M(K)$.

Lemma 4.4. If N and K are subsemimodules of an R-semimodule M such that $N \subseteq K$, then $c.rad_K(N) \subseteq c.rad_M(N)$.

Proof. If there is no classical prime subsemimodule of M containing N, then $c.rad_M(N) = M$. Since $c.rad_K(N)$ is a subsemimodule of K, we have $c.rad_K(N) \subseteq K \subseteq M = c.rad_M(N)$. There exists a classical prime subsemimodules of M containing N, then $c.rad_M(N)$ is a subsemimodule of M with $N \subseteq c.rad_M(N)$. Let W be a classical prime subsemimodule of M containing N. We have 2 cases to consider; $W \subseteq K$ and $W \nsubseteq K$.

Case 1. $W \subseteq K$. Then W is a classical prime subsemimodules of K containing N, so $c.rad_K(N) \subseteq W$.

Case 2. $W \nsubseteq K$. It is clear that $K \cap W$ is a classical prime subsemimodule of K. Since $N \subseteq K \cap W$, we have $c.rad_K(N) \subseteq K \cap W \subseteq W$. Hence, $c.rad_K(N) \subseteq c.rad_M(N)$.

Theorem 4.5. Let N be a proper subsemimodule of an R-semimodule M. Then

$$c.rad_M(N) \subseteq \bigcap \{(N,P) \mid P \text{ is a prime ideal of } R\}.$$

Proof. Let $B = \bigcap \{(N,P) \mid P \text{ is a prime ideal of } R\}$. We will show that $c.rad_M(N) \subseteq B$. Let $m \in c.rad_M(N)$ and let $(N,P) \in B$. Then by Lemma 3.3, we have $(N,P) = M$ or (N,P) is a classical prime subsemimodule of M.

Case 1. If $(N,P) = M$, then it is trivial that $m \in (N,P)$.

Case 2. If (N,P) is a prime subsemimodule of M, then (N,P) is a classical prime subsemimodule of M and $N \subseteq PM + N \subseteq (N,P)$, so $m \in (N,P)$. Therefore $c.rad_M(N) \subseteq \bigcap\{(N,P) \mid P$ is a prime ideal of $R\}$.

Theorem 4.6. Let N be a proper subsemimodule of an R-semimodule M. Then $rad_M(N) = \bigcap\{(N,P) \mid P$ is a prime ideal of $R\}$.

Proof. Let $B = \bigcap\{(N,P) \mid P$ is a prime ideal of $R\}$. We will show that $B \subseteq rad_M(N)$. Let L be a prime subsemimodule of M containing N and let $m \in B$. Since L is a prime subsemimodule of M, we have $(L:M)$ is prime ideal of R. There exists c with $c \in R - (L:M)$ such that $cm \in (L:M)M + N$. Therefore $cm = hs + k$, where $h \in (L:M), s \in M$ and $k \in N$. Since $hM \subseteq L$ and $N \subseteq L$, we have $hs \in L$ and $k \in L$. Then $cm \in L$. Since cM Ú L and L is a prime subsemimodule of $M, m \in L$. It follows $m \in rad_M(N)$ and hence $B \subseteq rad_M(N)$. Next, we will show that $rad_M(N) \subseteq B$. Let $m \in rad_M(N)$ and let $(N,P) \in B$. Then by Lemma 3.3, we have $(N,P) = M$ or (N,P) is a prime subsemimodule of M.

Case 1. If $\left(N,P\right) = M$, then it is trivial that $m \in (N,P)$.

Case 2. If (N,P) is a prime subsemimodule of M and M and $N \subseteq PM + N \subseteq (N,P)$, then $m \in (N,P)$. Therefore $B \subseteq rad_M(N)$ and hence $prad_M(N) = \bigcap\{(N,P) \mid P$ is a prime ideal of $R\}$.

Theorem 4.7. Let N be a proper subsemimodule of an R-semimodule M. Then $c.rad_M(N) \subseteq rad_M(N)$.

Proof. It follows from Theorem 4.5 and Lemma 4.6.

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