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Additive Properties of Measurable Set for Difference Two Measurable Set

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ABSTRAC: ***This paper will carry out the problem from [7] to be related difference of two measurable set. The problem is to prove the theorem if A and B are measurable sets such that $B \subset A$ and $m(B) < \infty$ then m(A – B) = m(A) – m(B). theorem proving is done through the study of properties measurable set.***

Keywords--- Aditif, Meaurable, difference two measurable set

1. INTRODUCTION

The development of modern measurement theory is characterized by the introduction of the concept outer measure by Henry Lebesgue in 1940. At that time outer measure is defined as the infimum of the total length of the intervals that cover the set ([7, p 55). With outer measure are introduced, has many problems that can be solved. The examples if the set is interval then it's outer measure is equal with interval length, but theoretically outer measure to have a weakness, because the outer measure do not meet the additive properties that $m*(A \cup B) \neq m*(A) + m*(B)$. That's why the researchers tried to cover up the weakness of the outer measure. Among researchers it is Henry Lebesgue, which defines the measure by using the concept outer measure.

By using the concept of measure, important issues that exist in the analysis can be developed such as in ([4], p313) about the properties of the open set, that the union of an arbitrary collection of open subsets in R is open in R, and on another litertur is ([5], p136) *if G_1 and G_2 are open subset of R, then $G_1 \cap G_2$ also open*. By using the concept of the measure, ([6], p20) give generalization abaut union of measurable set, that is the union of a sequence of measurable sets is measurable. Even problems in Real Analysis is not applicable, by using the concept of the measure, the problem can be proven to be valid. The example that *if A and B are open sets in R, then A - B is not necessarily open set in R,* using the concept of measure can be shown that *if A and B are measurable sets then A - B is measurable*.

Discussion outer measure of a set associated with the power set of the set, such as defining the outer measure of ([1] as follows, v will designate a finite-valued, finitely subadditive outer measure defined on the power set P(X) of a nonempty set X. ρ will designate the associated set function $\rho(E) = v(E) - v(E')$, where $E \subset X$. defining the outer measure can also define from the length of open interval as in the following definition. Suppose F is a collection of countable open intervals. For any J ∈ F, the total $\sum_{I\in J} l(I)$ is a positive real number. Let E be any set, take a subset C of F to C is a collection of J from open intervals $\{I_i\}$ such that $E \subset \bigcup_i I_i$. If the set C is written C = {J: J ⊂ F and J cover E}. Outer measure m * (E) of the set E is m*(E) = inf $\{\sum_i l(I_i) : \{I_i\}$ open interval and $E \subset \bigcup_i I_i.\}$ ([7], p55).

Defining the measure set by [3*] if v is outer measure, then S_v the v-measurable sets = $\{E \subset X / v(G) = v(G\cap E) + v(G\cap E')\}$ for all $G \subset X$.* Definition of the measurable set, according to [7] is as follows, the set E is said measurable, *if $\forall A \subset R$, apply $m*(A) = m*(A\cap E) + m*(A\cap E^c)$*, and if E is measurable set, then *$m*(E) = m(E)$*. Meanwhile, according to ([2], p115), *if A and B are measurable sets, then $m(A\cup B) = m(A) + m(B) - m(A\cap B)$*. This means that *$m(A \cup B) \leq m(A) + m(B)$ for an measure set A and B,*

This paper will carry out the problem from ([7], p68) to be related difference of two measurable set. The problem is to prove the theorem *if A and B are measurable sets such that $B \subset A$ and $m(B) < \infty$ then m(A – B) = m(A) – m(B).* theorem proving is done through the study of properties measurable set.

2. PROPERTIES OF MEASURABLE SET

Theorem 1. *If E is measurable set then E^c is measurable set.*

Proof: Because E is measurable set, by definition $\forall A \subset R$, we have $m*(A) = m*(A\cap E) + m*(A\cap E^c) = m*(A\cap E^c) + m*(A\cap E) = m*(A\cap E^c) + m*(A\cap (E^c)^c)$

So E^c is measurable set.

Theorem 2 *if D and E measurable set, then D∩E measurable*

Proof: Because D is measurable set, by definition, $\forall A \subset R$,

We have $m*(A) = m*(A\cap D) + m*(A\cap D^c)$.

$= m*((A\cap D) \cap E) + m*((A\cap D)\cap E^c) + m*(A\cap D^c)$.

$= m*(A\cap(D\cap E)) + m*(A\cap D^c) + m*(A\cap(D\cap E^c))$

$\geq m*(A\cap(D\cap E)) + m*(A\cap(D^c \cup E^c)$

Because A$\cap$ $(D^c \cup E^c)$ = (A$\cap$ D^c) $\cup$ $(A \cap (D \cap E^c)$
= m*(A$\cap$(D$\cap$E)) + m*(A$\cap$ $(D \cap E)^c$)
then m*(A) ≥ m*(A$\cap$(D$\cap$E)) + m*(A$\cap$ $(D \cap E)^c$) ………………………..1)
Because A = (A$\cap$ $(D \cap E)$) $\cup$ $(A \cap (D \cap E)^c)$, then
m*(A) ≤ m*((A$\cap$ $(D \cap E)$ + m*$(A \cap (D \cap E)^c)$………………………..2)
from 1) and 2) we have m*(A) = m*((A$\cap$ $(D \cap E)$ + m*$(A \cap (D \cap E)^c)$
So D$\cap$ E measurable

Theorem 3 If $\{E_1, E_2, \ldots, E_n\}$ are finite collection of measurable set, then $\cup_{i=1}^{n} E_i$ is measurable

Proof : We use induction of n, as below;

- Show true for n = 1
  for $n = 1$ we have $\cup_{i=1}^{1} E_1$= E_1, because E_1 measurable, then $\cup_{i=1}^{1} E_1$ measurable (true)
- Assume true for n = k-1
  for $n = k - 1$ we have $\cup_{i=1}^{k-1} E_i$ measurable
- It will be proved $\cup_{i=1}^{k} E_i$ measurable
  $\cup_{i=1}^{k} E_i = E_1 \cup E_2 \cup \ldots \cup E_{k-1} \cup E_k$
  $= \cup_{i=1}^{k-1} E_i \cup E_k$
  Because $\cup_{i=1}^{k-1} E_i$ measurable, and E_k measurable, then
  $\cup_{i=1}^{k-1} E_i \cup E_k = \cup_{i=1}^{k} E_i$ measurable.

Theorem 4 If $\{E_1, E_2, \ldots, E_n\}$ are finite collection of measurable set, then $\cap_{i=1}^{n} E_i$ is measurable

Proof: We use induction of n, as below;

- Show true for n = 1
  for $n = 1$ we have $\cap_{i=1}^{1} E_1$ = E_1 because E_1 measurable, then $\cap_{i=1}^{1} E_1$ measurable (true)
- Assume true for n = k-1
  for $n = k - 1$ we have $\cap_{i=1}^{k-1} E_i$ measurable
- It will be proved $\cap_{i=1}^{k} E_i$ measurable
  $\cap_{i=1}^{k} E_i = E_1 \cap E_2 \cap \ldots \cap E_{k-1} \cap E_k$
  $= \cap_{i=1}^{k-1} E_i \cap E_k$
  Because $\cap_{i=1}^{k-1} E_i$ measurable, and E_k measurable, then $\cap_{i=1}^{k-1} E_i \cap E_k = \cap_{i=1}^{k} E_i$ measurable.

Teorema 5 Let $E_1, E_2, E_3, \ldots, E_n$ are finite sequance from disjoint measurable sets, then for arbitrary set A.

$$m^*\left(A \bigcap \left[\bigcup_{I=1}^{n} E_i\right]\right) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

Proof: We use induction,

1) for $n = 1$ thru, because $m^*(A \bigcap [\bigcup_{I=1}^{1} E_i]) = m^*(A \cap E_1) = \sum_{i=1}^{1} m^*(A \cap E_i)$
2) Assume the statement is true for $n = k - 1$ with $1 < k \le n$, then
$$m^*(A \bigcap [\bigcup_{I=1}^{k-1} E_i]) = \sum_{i=1}^{k-1} m^*(A \cap E_i) \qquad (1)$$
3) It will be proved that the equation is also true for $n = k$ with $1 < k \le n$.
Consider the equation (1). With added $m^*(A \cap E_k)$ for equation (1) at two side,
$$m^*\left(A \bigcap \left[\bigcup_{I=1}^{k-1} E_i\right]\right) + m^*(A \cap E_k) = \sum_{i=1}^{k-1} m^*(A \cap E_i) + m^*(A \cap E_k)$$
$$= \sum_{i=1}^{k} m^*(A \cap E_i) \qquad (2)$$
Because $E_1, E_2, E_3, \ldots, E_n$ disjoint, then $\bigcup_{i=1}^{k-1} E_i$ with E_k disjoint. Then
$*$ $\bigcup_{i=1}^{k-1} E_i \cap E_k = \emptyset$ (3)
$*$ $\bigcup_{i=1}^{k-1} E_i \cap E_k^c = \bigcup_{i=1}^{k-1} E_i$ (4)
Consider
$*$ $\bigcup_{i=1}^{k} E_i \cap E_k = \left(\bigcup_{i=1}^{k-1} E_i \cup E_k\right) \cap E_k = \left(\bigcup_{i=1}^{k-1} E_i \cap E_k\right) \cup (E_k \cap E_k)$ (5)
$*$ $\bigcup_{i=1}^{k} E_i \cap E_k^c = \left(\bigcup_{i=1}^{k-1} E_i \cup E_k\right) \cap E_k^c = \left(\bigcup_{i=1}^{k-1} E_i \cap E_k^c\right) \cup (E_k \cap E_k^c) = \bigcup_{i=1}^{k-1} E_i \cap E_k^c$ (6)
form (3) with (5) and (4) with (6) we have
$\bigcup_{i=1}^{k} E_i \cap E_k = E_k$ dan $\bigcup_{i=1}^{k} E_i \cap E_k^c = \bigcup_{i=1}^{k-1} E_i$
Then, equation (2) can write
$$m^*(A \bigcap [\bigcup_{I=1}^{k} E_i] \cap E_k^c) + m^*(A \cap [\bigcup_{I=1}^{k} E_i] \cap E_k) = \sum_{i=1}^{k} m^*(A \cap E_i) \qquad (7)$$
Because $E_1, E_2, E_3, \ldots, E_n$ measurables, then E_k measurable. From definition, and take, set test $A \cap [\bigcup_{I=1}^{k} E_i]$, then
$$m^*(A \bigcap [\bigcup_{I=1}^{k} E_i] \cap E_k^c) + m^*(A \cap [\bigcup_{I=1}^{k} E_i] \cap E_k) = m^*(A \cap [\bigcup_{I=1}^{k} E_i]) \qquad (8)$$
from (7) and (8) then

$$m^*\left(A \cap \left[\bigcup_{I=1}^{k} E_i\right]\right) = \sum_{i=1}^{k} m^*(A \cap E_i)$$

So, base from induction principle, if $E_1, E_2, E_3, \ldots, E_n$ are finite sequance from disjoint measurable sets, then for arbitrary set A, apply

$$\mathrm{m}^*\left(\mathrm{A} \bigcap \left[\bigcup_{\mathrm{I}=1}^{\mathrm{n}} \mathrm{E_i}\right]\right) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{m}^*(\mathrm{A} \cap \mathrm{E_i})$$

Theorem 6 If $E_1, E_2, E_3, \ldots, E_n$ are finite sequence of disjoint measurable set, then $m(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m(Ei)$

Proof: Because $\{E_1, E_2, E_3, \ldots, E_n\}$ measurable and disjoint, then from theorem 3 $\bigcup_{i=1}^{n} \mathrm{E}_i$ measurable. Take $= \mathbb{R}$, then

$m(\bigcup_{i=1}^{n} \mathrm{E}_i)$ $= m^*(\bigcup_{i=1}^{n} \mathrm{E}_i)$ (outher measure equal with measure)

$= m^*(\mathbb{R} \cap (\bigcup_{i=1}^{n} \mathrm{E}_i))$ (Because $\bigcup_{i=1}^{n} \mathrm{E}_i = \mathbb{R} \cap \bigcup_{i=1}^{n} \mathrm{E}_i$)

$= \sum_{i=1}^{n} m^*(\mathbb{R} \cap \mathrm{E}_i)$ ($\mathrm{E}_1, \mathrm{E}_2, \ldots, \mathrm{E}_n$ finite sequence of disjoint measurable set)

$= \sum_{i=1}^{n} m^*(\mathrm{E}_i)$ (Because $\mathbb{R} \cap \mathrm{E}_i = \mathrm{E}_i$)

$= \sum_{i=1}^{n} m(\mathrm{E}_i)$ (outher measure equal with measure)

So, If $E_1, E_2, E_3, \ldots, E_n$ are finite sequence of disjoint measurable set, then $m(\bigcup_{i=1}^{n} \mathrm{E}_i) = \sum_{i=1}^{n} m(\mathrm{E}i)$

3. PROOF THE PROBLEM

Theorem 7 If E_1 and E_2 are measurable sets, such that $E_2 \subset E_1$ and $m(E_2) < \infty$, then $m(E_1 – E_2) = m(E_1) – m(E_2)$.

Proof: Consider that: $E_1 – E_2 = E_1 \cap E_2^c$

Let: $E_1 = E_1 \cap R$

$= E_1 \cap (E_2^c \cup E_2)$

$= (E_1 \cap E_2^c) \cup (E_1 \cap E_2)$ Because $E_2 \subset E_1$

$= (E_1 \cap E_2^c) \cup E_2$

Because E_1 and E_2 are measurable sets, base from theorem , then E_2^c is measurable.

Base theorem 2 because E_1 and E_2^c are measurable, then $E_1 \cap E_2^c$ is measurable.

$E_1 = (E_1 \cap E_2^c) \cup E_2$

Claim: $(E_1 \cap E_2^c) \cap E_2 = \emptyset$

proof claim: $(E_1 \cap E_2^c) \cap E_2 = (E_1 \cap E_2) \cap (E_2^c \cap E_2) = E_2 \cap \emptyset = \emptyset$

Because $(E_1 \cap E_2^c) \cap E_2 = \emptyset$ dan $E_1 = (E_1 \cap E_2^c) \cup E_2$

Base theorem 6 we have

$m (E_i) = m ((E_1 \cap E_2^c) \cup (E_2))$

$m (E_1) = m (E_1 \cap E_2^c) + m (E_2)$

$m (E_1) = m (E_1 - E_2) + m (E_2)$ karena $E_1 – E_2 = E_1 \cap E_2^c$

$m (E_1) - m (E_2) = m (E_1 - E_2)$.

So, $m (E_1 - E_2) = m (E_1) - m (E_2)$.

4. CONCLUTION

A new properties of measurable set have been discovered recently. Properties of measurable set have attracted researchers of the field to investigate these newly discovered properties in detail. This article investigate the properties of two measurable set that $m(E_1 – E_2) = m(E_1) – m(E_2)$.

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