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An Order Rank Method for Multi-Objective Linear Programming

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ABSTRACT—*Inthis paper, a method is proposed to solve the Multi- objective linear programming problem (MOLPP);the method uses the idea of an order relation to rank the interval numbers.It checks the dominance of the optimal objective values,which are assumed to be an interval numbers by comparing order relations based on their mid-points and radius.Moreover, we demonstrate some numerical examples which show that,the proposed method could be a worthy alternative.*

Keywords—MOLPP, Pareto-Optimal, Order Rank Method,

1. INTRODUCTION

When more than one concern exists, or several numbers of objectives are to be optimised, the idea of multi-objective optimization is needed. Multi-objective optimization also known as multi-criteria optimization or pareto-optimization is an area of multiple criteria decision-making, which involves with mathematical optimization problems concerning more than one objective function to be optimisedsimultaneously [6]. However, it has been applied in many fields of sciences, engineering, business, economics and logistics [4-5].Many researchers have studied the pareto-optimal solution of MOLPP,the idea of weighted sum method can be found in Zadeh [11]. Koski [8] applied the weighted sum method to structural optimization, Marglin [9] developed the ε-constraints method, the adaptive weighted sum (AWS) method was recently developed by [7] to addressed sum of the drawbacks of WSM. In 2009, Arsham et al. [1] proposed a solution algorithm to LP problems-improved algebraic method (IAM) - which reduces considerably the computational complexity because it works directly on the decision variables, in that no slack, surplus or artificial variables are introduced.

In this paper, a simple methodology of the approximation of the pareto-optimal solution called order rank method (ORM) is presented, which does notrequire the introduction of additional inputs from the decision maker (DM).

2. MULTI-OBJECTIVE LINEAR PROGRAMMING: EXISTING METHODS

A MOLPP simultaneously optimisesan n-objective subject to the given constraints [4]. Normally, the problem has no optimal solution that could optimize all objectives simultaneously. The concept of optimal solution gives rise to the notion of non-dominated solutions, for which no improvement in any objective function is possible without sacrificing at least one of the objective functions [6]. The n-objectives LPP is formulated as follows:

Problem: "Max" or "Min"
$$\left.\begin{array}{l} z_1 = c_1 x \\ z_2 = c_2 x \\ \vdots \quad \vdots \\ z_n = c_n x \end{array}\right\} \quad (1)$$

Subject to: $x \in S = \{x \in R^n : Ax = b,\ x \geq 0\}$. (2)

where x an n-dimensional vector of the decision variables, $z_1, z_2, \ldots, z_n$ are the n-distinct linear objective functions of thedecision vector, and $c_1, c_2, \ldots, c_n$ denote the n-dimensional cost vectors. A is an $m \times n$ constraints matrix, and b represents m-dimensional constant vector.Furthermore, an optimal solution to this problem is called; pareto-optimal and is defined precisely as follows:

Definition 2.1:A solution is called pareto-optimal (or alternatively; efficient, non-dominated or non-inferior), if there is noother feasible solution that is equal or better with respect toall objectives included in the model.

In this study, to explain the multi-objective solutions, we utilizedwidely used in practice. the two well-known multi-objectives methods .The traditional methods are; the weighted sum method (WSM) and ε-constraints method.

2.1 Weighted Sum Method (WSM)

Zadeh suggested the weighted sum approach in 1963[11]. The idea behind this method is to associate each objective function with a weighting coefficient and maximize or minimize the weighted sum of the objectives [11].That means the multiple objectives functions are transformed into a single objective function and solved as a single objectiveoptimization problem [4]. The modified problemcan be represented as

$$F(x) = \sum_{i=0}^{k} w_i f_i(x) \qquad (3)$$

where $f_i(x)$ is the ith objective function, k is the number of objectives, $w_i \geq 0$ and $\sum_{i=0}^{k} w_i = 1$. The problem with thisapproach is that the solutions may vary significantly as the weighting coefficient change,and also the optimal solutiondistribution is not unique as stated in [4] and [7].

2.2 ε-Constraints Method

Constraints method dates back to Marglin 1967[8]. In this method, one of the objective functions is selected to be optimized, and all other objective functions are transformed into constrained by setting an upper bound to each of them [8].For any given of set of right-hand side (RHS) values, the problem is but standard LPP. After the problem is solved for one set of achievement level,their values are modified by the decision maker and the problem is solved again with a different set of (RHS) values [4].This process is repeated until a solution is found that is acceptable to the decision maker.

The problem to be solve is now of the form

Problem: "Max" or "Min": $f_l(x)$

Subject to: $f_j(x) \leq \varepsilon_j \ x \in S$ where $l \in \{1,2,\ldots,l\}$. (4)

3. INTERVAL ARITHMETIC

In this section, some basic and important definitions in the study of interval numbers are reviewed.These definitions will help in presenting our algorithms.

Definition 3.1: A real interval vector $U \in I(R^n)$ is a set of the form $\underline{U} = (\underline{U})_{n\times n}$,where $i = 1,2,\ldots,n$ and $\underline{U_i} = [u_i^L, u_i^R] \in I(R^n)$.

Definition 4.2: A real interval matrix $A \in I(M(R^n))$ is a set of theform $\underline{A} = (a_{ij})_{n\times n}$ where $i = 1,2,\ldots,n$and $\underline{a_{ij}} = [a_{ij}^L, a_{ij}^R] \in I(R^n)$.

Let $\underline{A} = [a^L, a^R]$ and $\underline{B} = [b^L, b^R]$, then

1. $\underline{A} + \underline{B} = [a^L + b^L, a^R + b^R]$ (Addition)

2. $\underline{A}-\underline{B}=[a^L-b^L, a^R-b^R]$ (Subtraction)

3. $\underline{A}\cdot\underline{B}=[\min\{a^Lb^L, a^Lb^R, a^Rb^L, a^Rb^R\}, \max\{a^Lb^L, a^Lb^R, a^Rb^L, a^Rb^R\}]$ (Multiplication)

4. $\dfrac{\underline{A}}{\underline{B}}=[a^L, a^R][\dfrac{1}{b^L}, \dfrac{1}{b^R}]$ for $0\notin b$ (Division)

4. AN ORDER RANK METHOD (ORM)

For a given multi-objective linear programming problem, the IAM used to find the coordinates of the corner points of the feasible region [1].Parametric representation $f_1(\lambda_i), f_2(\lambda_i),..., f_n(\lambda_i)$ of the feasible region is then developed using parameters $\lambda_1, \lambda_2,...,\lambda_n$, $\forall\lambda_i^{'}s\geq 0$ such that $\sum_{i=1}^{n}\lambda_i=1$, where n denotes the number of vertices.

Proposition 4.1

The Maximum or (Minimum) points of a (MOLPP) with a bounded feasible region correspond to the Maximization or (Minimization) of the parametric objective function $f_1(\lambda_i), f_2(\lambda_i),..., f_n(\lambda_i)$ for $i=1,2,\dots,n$.

Proof:

Following [1], let the terms with the largest (smallest) coefficients in $f_1(\lambda_i), f_2(\lambda_i),..., f_n(\lambda_i)$ with their corresponding parameters $\lambda_{L1}, \lambda_{L2},...,\lambda_{Ln}$, and $\lambda_{S1}, \lambda_{S2},...,\lambda_{Sn}$, be denoted by λ_{Lh} and λ_{Sh} respectively.Since, each $f_i(\lambda_i)$ such that $i=1,2,\dots,n$ are (linear) convex combination of the coefficient,the optimal solution of $f_i(\lambda_i)$ are obtained by setting both λ_{Lh} or λ_{Sh} equal to 1and all other $\lambda_{Li}{'}s$ and $\lambda_{Si}{'}s$ to zero,where $h,k\in\{1,2,\dots,n\}$. and the proof is completed. ■

Suppose that the second parametric representation of the optimal solution is of the form:

$$g_1(\lambda_i)=\alpha_{11}\lambda_1+\alpha_{12}+\cdots+\alpha_{1n}\lambda_n$$
$$g_2(\lambda_i)=\alpha_{21}\lambda_1+\alpha_{22}+\cdots+\alpha_{2n}\lambda_n$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$g_n(\lambda_i)=\alpha_{n1}\lambda_1+\alpha_{n2}+\cdots+\alpha_{nn}\lambda_n$$

We reduce the system into an augmented matrix

$$\begin{pmatrix} g_1\\ g_2\\ \vdots\\ g_n\end{pmatrix}=\begin{pmatrix}\alpha_{11} & \alpha_{12}\cdots & \alpha_{1n}\\ \alpha_{12} & \ddots & \alpha_{2n}\\ \vdots & \ddots & \vdots\\ \alpha_{n1} & \cdots & \alpha_{nn}\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\\ \vdots\\ \lambda_n\end{pmatrix} \qquad (5)$$

Setting $[\lambda_1, \lambda_2,...,\lambda_n]=[1,0,0,...,0]$ in (5), we have

$$A_1=\begin{pmatrix}\alpha_{11}\\ \alpha_{21}\\ \vdots\\ \alpha_{n1}\end{pmatrix} \qquad (6)$$

Setting $[\lambda_1, \lambda_2,...,\lambda_n]=[0,1,0,...,0]$ in (5), we have

$$A_2=\begin{pmatrix}\alpha_{12}\\ \alpha_{22}\\ \vdots\\ \alpha_{n2}\end{pmatrix} \qquad (7)$$

$\vdots$

$\vdots$

Setting $[\lambda_1, \lambda_2, \ldots, \lambda_n] = [0,0,0,\ldots,1]$ in (5), we have

$$A_n = \begin{pmatrix} \alpha_{1n} \\ \alpha_{2n} \\ \vdots \\ \alpha_{nn} \end{pmatrix} \tag{8}$$

Let us consider $A_i = [\alpha_{ij}, \alpha_{i+1j}, \ldots, \alpha_{nj}]^T$ as an optimal objective values for $i = 1,2,\ldots,n$ and $j \geq 1$. Without loss of generality, we can define an interval number $\underline{A}_i = [\min\{A_i\}, \max\{A_i\}]$ or $\underline{A}_i = [\alpha_{ij}^L, \alpha_{ij}^R]^T$, where superscript L and R denote the lower and upper bounds of the interval number respectively.
The mid-point and radius of the interval numbers are defined as follows

$$m(\underline{A}_i) = \frac{\min\{A_i\} + \max\{A_i\}}{2} = \frac{\alpha_{ij}^L + \alpha_{ij}^R}{2} \tag{9}$$

$$w(\underline{A}_i) = \frac{\max\{A_i\} - \min\{A_i\}}{2} = \frac{\alpha_{ij}^R - \alpha_{ij}^L}{2} \tag{10}$$

If the feasible regions of MOLPP are available, it is simple to get thepareto-optimal solution if thefollowing steps are considered.

Algorithm 4.1
Step 1: Form a parametric representation of the feasible regions to obtain the optimal solution.
Step 2: Form a second parametric representation of the optimal solutions.
Step 3: Form an augmented matrix of the second parametric representation and apply the setting as the basis of the parameters as shown in equation (6), (7) and so on.
Step 4: Check the Dominance: Given $\underline{A}_1, \underline{A}_2, \underline{A}_3, \ldots, \underline{A}_n$ as a sets of optimal objective values,

- *Dominance for Maximization problem*

$\underline{A}_1$ Dominates $\underline{A}_2, \underline{A}_3, \underline{A}_4, \ldots, \underline{A}_n$,
If $m(\underline{A}_1) \leq m(\underline{A}_2), m(\underline{A}_3), \ldots, m(\underline{A}_n)$ and $w(\underline{A}_1) \geq w(\underline{A}_2), w(\underline{A}_3), \ldots, w(\underline{A}_n)$. .

- *Dominance for Minimization problem*

$\underline{A}_1$ Dominates $\underline{A}_2, \underline{A}_3, \underline{A}_4, \ldots, \underline{A}_n$,
If $m(\underline{A}_1) \geq m(\underline{A}_2), m(\underline{A}_3), \ldots, m(\underline{A}_n)$ and $w(\underline{A}_1) \leq w(\underline{A}_2), w(\underline{A}_3), \ldots, w(\underline{A}_n)$.
Step 5: Conclude that x^* with its corresponding objective value $\underline{A}_1$ is the pareto-optimal solution.

5. NUMERICAL EXAMPLE

Consider the MOLPP below:

$$\text{Max: } \begin{cases} z_1 = -2x_1 + 3x_2 \\ z_2 = 3x_1 - x_2 \\ z_3 = x_1 + 2x_2 \end{cases} \tag{11}$$

Subject to:

$$\left.\begin{array}{ll} (i) & -x_1 + x_2 \le 2 \\ (ii) & x_1 + 2x_2 \le 6 \\ (iii) & x_1 \le 4 \\ (iv) & x_1 + x_2 \ge 1 \\ (v) & x_1 \ge 0 \\ (vi) & x_2 \ge 0 \end{array}\right\} \qquad (12)$$

The table below shows the solutions of this problem solved by IAM, where under feasibility test 1 and 0 represent "Yes" and "No" respectively.

Table 1: Solution

No.	x_1	x_2	Feasible?	Equation
1.	2/3	8/3	1	(i) and (ii)
2.	4	6	0	(i) and (iii)
3.	-1/2	3/2	0	(i) and (iv)
4.	0	2	1	(i) and (v)
5.	-2	0	0	(i) and (vi)
6.	4	1	1	(ii) and (iii)
7.	-4	5	0	(ii) and (iv)
8.	0	3	0	(ii) and (v)
9.	6	0	0	(ii) and (vi)
10.	4	-3	0	(iii) and (iv)
11.	[0,4]	-	0	(iii) and (v)
12.	4	0	1	(iii) and (vi)
13.	0	1	1	(iv) and (v)
14.	1	0	1	(iv) and (vi)
15.	0	0	0	(v) and (vi)

Table 2: Feasible Solutions

x_1	$\frac{2}{3}$	0	4	4	0	1
x_2	$\frac{8}{3}$	2	1	0	1	0

Parametric representation of the feasible region is then developed using $\lambda_1, \lambda_2, \ldots, \lambda_6$, $\forall \lambda_i^{'} s \ge 0$ such that $\sum_{i=1}^{6} \lambda_i = 1$.

$$x_1 = \frac{2}{3}\lambda_1 + 4\lambda_3 + 4\lambda_4 + \lambda_6 \qquad (13)$$

$$x_2 = \frac{8}{3}\lambda_1 + 2\lambda_2 + \lambda_3 + \lambda_5 \quad (14)$$

Substituting (13) and (14) into the objective function z_1, we have

$$z_1 = \frac{20}{3}\lambda_1 + 6\lambda_2 - 5\lambda_3 - 8\lambda_4 + 3\lambda_5 - 2\lambda_6$$

Setting $\lambda_1 = 1$ and all other $\lambda_i{'}s$ to zero, the maximum value found to be $\frac{20}{3}$.

Substituting (13) and (14) into the objective function z_2, we have

$$z_2 = -\frac{2}{3}\lambda_1 - 2\lambda_2 + 11\lambda_3 + 12\lambda_4 - \lambda_5 + 3\lambda_6$$

Setting $\lambda_4 = 1$ and all other $\lambda_i{'}s$ to zero, the maximum value found to be12.

Substituting (13) and (14) into the objective function z_3, we have

$$z_3 = 6\lambda_1 + 4\lambda_2 + 6\lambda_3 + 4\lambda_4 + 2\lambda_5 + \lambda_6$$

Setting $\lambda_3 = 1$ and all other $\lambda_i's$ to zero, the maximum value found to be 6.
Now the optimal solutions are

Table 3: Optimal solutions

$x_1 = \frac{2}{3}$ $x_2 = \frac{8}{3}$	$x_1 = 4$ $x_2 = 1$	$x_1 = 4$ $x_2 = 0$

We form the second parametric representation of the optimal solution, for $\lambda_1,\ldots,\lambda_3$, $\forall \lambda_i's \geq 0$ such that $\sum_{i=1}^{3} \lambda_i = 1$.

$$x_1 = \frac{2}{3}\lambda_1 + 4\lambda_2 + 4\lambda_3 \quad (15)$$

$$x_2 = \frac{8}{3}\lambda_1 + \lambda_2 + 0\lambda_3 \quad (16)$$

Substituting (15) and (16) into the objective functions (11), we have

$$z_1 = \frac{20}{3}\lambda_1 - 5\lambda_2 - 8\lambda_3$$

$$z_2 = -\frac{2}{3}\lambda_1 + 11\lambda_2 + 12\lambda_3$$

$$z_3 = 6\lambda_1 + 6\lambda_2 + 4\lambda_3$$

An augmented matrix of the above system is now form

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \frac{20}{3} & -5 & -8 \\ \frac{-2}{3} & 11 & 12 \\ 6 & 6 & 4 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \quad (17)$$

Setting $[\lambda_1, \lambda_2, \lambda_3] = [1,0,0]$ in (17), we have $A_1 = \begin{pmatrix} \frac{20}{3} \\ \frac{-2}{3} \\ 6 \end{pmatrix}$

Setting $[\lambda_1, \lambda_2, \lambda_3] = [0,1,0]$ in (17), we have $A_2 = \begin{pmatrix} -5 \\ 11 \\ 6 \end{pmatrix}$

Setting $[\lambda_1, \lambda_2, \lambda_3] = [0,0,1]$ in (17), we have $A_3 = \begin{pmatrix} -8 \\ 12 \\ 4 \end{pmatrix}$

Now, $\underline{A}_1 = [-\frac{2}{3}, \frac{20}{3}]$, $\underline{A}_2 = [-5,11]$ $\underline{A}_3 = [-8,12]$
We compute the mid-point and radius as follows:

$m(\underline{A}_1) = 3;$ $m(\underline{A}_2) = 3;$ $m(\underline{A}_3) = 2.$
$w(\underline{A}_1) = 3.7;$ $w(\underline{A}_2) = 8;$ $w(\underline{A}_3) = 10.$

It is clear from the above that, $m(\underline{A}_3) \leq m(\underline{A}_1), m(\underline{A}_2)$ and also $w(\underline{A}_3) \geq w(\underline{A}_2), w(\underline{A}_3)$

Therefore, $\underline{A}_3$ dominates both $\underline{A}_1$ and $\underline{A}_2$.
Hence, we conclude that $x = (4,0)$ with optimal objective value $\underline{A}_3 = [-8,12,4]$ is the pareto-optimal solution.

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