Weakly Prime and Weakly Semiprime Left ideal in Ternary Semigroups

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ABSTRACT— ***The purpose of this paper is to introduce the notion of a weakly prime ideals in ternary semigroups, we study left ideals, weakly prime and weakly semiprime ideals in ternary semigroups. Some characterizations of*** Γ***-weakly semiprime and weakly prime ideals are obtained.***

Keywords— ternary semigroup, weakly semiprime, weakly prime, m-Systems, n-Systems

1. INTRODUCTION

There is a large literature dealing with ternary operations. The notion of ternary semigroup is a natural generalization of ternary group. Algebraic structures play a prominent role in mathematics with wide applications in many disciplines such as computer sciences, information sciences, engineering, physics etc. Banach find some applications in ternary semigroup. He gave an example to show that a ternary semigroup is not necessarily reduce to an ordinary semigroup. Los [9] studied some properties of ternary semigroup and proved that every ternary semigroup can be embedded in a semigroup.

In 1932, Lehmer introduced the concept of ternary semigroup [8]. The algebraic structures of ternary semigroups were also studied by some authors, for example, Sioson studied ideals in ternary semigroups [17]. A non-empty set S is called a ternary semigroup if there exists a ternary operation $[]: S\times S\times S \to S;$ satisfying the following identity, for any $x_1,\ x_2,\ x_3,\ x_4,\ x_5 \in S,$

$$\left[\left[x_1x_2x_3\right]x_4\ x_5\ \right]=\left[\ x_1\left[\ x_2x_3x_4\right]x_5\right]=\left[x_1x_2\left[x_3\ x_4\ x_5\right]\right]$$

Any semigroup can be reduced to a ternary semigroup. However, Banach showed that a ternary semigroup does not necessarily reduce to a semigroup by this example. Let $S=\{-i,0-i\}$, where S is a ternary semigroup while S is not a semigroup under the multiplication over complex numbers. Los showed that every ternary semigroup can be embedded in a semigroup [9]. In this paper we characterize the ternary semigroups, we study left ideals, weakly prime and weakly semiprime ideals in ternary semigroups.

2. BASIC PROPERTIES

In this section we refer to [2, 3] for some elementary aspects and quote few definitions and examples which are essential to step up this study. For more detail we refer to the papers in the references.

Definition 2.1. [2, 3] A nonempty set S with a ternary operation $(x,y,z)\mapsto[xyz]$ satisfying the associative law, that is, for any $x_1,\ x_2,\ x_3,\ x_4,\ x_5 \in S,$

$$\left[\left[x_1x_2x_3\right]x_4\ x_5\right]=\left[\ x_1\left[\ x_2x_3x_4\right]x_5\right]=\left[x_1x_2\left[x_3\ x_4\ x_5\right]\right]$$

is called a ternary semigroup.

If A, B and C are nonempty subsets of a ternary semigroup S, then the product $[ABC]$ is the set of all elements $[abc] \in S$, where $a \in A, b \in B$ and $c \in C$, i.e.,

$$[ABC] = \{[abc] : a \in A,\ b \in B,\ c \in C\}$$

In dealing with singleton sets we write, for example, $[\{A\}BC]$ by $[aBC]$.

Definition 2.2. [2, 3] Let S be a ternary semigroup. A nonempty subset A of is called a left (respectively, right) ideal of S if $[SSA] \subseteq A$ (respectively, $[ASS] \subseteq A$). If A is both a left and a right ideal of S, then A is called a two sided ideal of S.

Definition 2.3. Let S be a ternary semigroup and A an ideal of S. Then A is said to be weakly prime if for all $a, b \in S, [aSb] \subseteq A$ implies $a \in A$ or $b \in A$.

Definition 2.4. Let S be a ternary semigroup and A an ideal of S. Then A is said to be weakly semiprime if for all $a, \in S, [aSa] \subseteq A$ implies $a \in A$.

Definition 2.5. Let S be a ternary semigroup and $\emptyset \neq A \subseteq S$. Then A is called an m-system of S if for each $a, b \in A$ there exist $c \in A$ and $x \in S$ such that $c = [axb]$.

Remark. Let S be a ternary semigroup and $\emptyset \neq A \subseteq S$. Then A is called an m-system of S if for any $a, b \in A$, there exists $c \in A$ such that $c \in [aSb]$.

Definition 2.6. Let S be a ternary semigroup and $\emptyset \neq A \subseteq S$. Then A is called an n-system of S if for each $a \in A$, there exist $c \in A$ and $x \in S$ such that $c = [axa]$.

Remark. Let S be a ternary semigroup and $\emptyset \neq A \subseteq S$. Then A is called an n-system of S if for any $a, b \in A$, there exists $c \in A$ such that $c \in [aSa]$.

3. LEFT IDEALS IN TERNARY SEMIGROUPS

The results of the following lemmas seem to play an important role to study ternary semigroups; these facts will be used frequently and normally we shall make no reference to this lemma.

Lemma 3.1. Let S be a ternary semigroup, and let A be a left ideal of S. Then $(A:a:b)_L$ is a left ideal in S, where $(A:a:b)_L = \{x \in S : [xab] \in A\}$.

Proof. Suppose that S is a ternary semigroup. Let $s, r \in S, x \in (A:a:b)_L$. Then $[xab] \in A$ and so that

$$[[srx]ab] = [sr[xab]] \in [SSA] \subseteq A$$

Therefore $[srx] \in (A:a:b)_L$ so that $[SS(A:a:b)_L] \subseteq (A:a:b)_L$. Hence $(A:a:b)_L$ is a left ideal in S.

Remark. Let S be a ternary semigroup and let A be a left ideal of S. It is easy to verify that $A \subseteq (A:a:b)_L$.

Corollary 3.2. Let S be a ternary semigroup, and let A be a right ideal of S. Then $(A:a:b)_R$ is a right ideal in S, where $(A:a:b)_R = \{x \in S : [abx] \in A\}$.

Proof. This follows from Lemma 3.1.

Remark. Let S be a ternary semigroup and let A be a right ideal of S. It is easy to verify that $A \subseteq (A:a:b)_R$.

Lemma 3.3. Let S be a ternary semigroup, and let A, B, C be left ideals of S. Then $(A:B:C)_L$ is a left ideal in S, where $(A:B:C)_L = \{x \in S : [xBC] \subseteq A\}$.

Proof. Suppose that S is a ternary semigroup. Let $x \in (A:B:C)_L$. Then $[xBC] \subseteq A$ so that

$$[[SSx]BC] = [SS[xBC]] \subseteq [SSA] \subseteq A$$

Therefore $[SSx] \subseteq (A:B:C)_L$ so that $[SS(A:B:C)_L] \subseteq (A:B:C)_L$. Hence $(A:B:C)_L$ is a left ideal in S.

Remark. Let S be a ternary semigroup and let A, B, C, D be left ideals of S. It is easy to verify that $(A:B:C)_L \subseteq (A:B:D)_L$, where $D \subseteq C$.

Corollary 3.4. Let S be a ternary semigroup, and let A, B, C be right ideals of S. Then $(A:B:C)_R$ is a right ideal in S, where $(A:B:C)_R = \{x \in S : [BCx] \subseteq A\}$.

Proof. This follows from Lemma 3.3.

Remark. Let S be a ternary semigroup and let A, B, C, D be right ideals of S. It is easy to verify that $(A:B:C)_R \subseteq (A:B:D)_R$, where $D \subseteq C$.

4. WEAKLY PRIME AND WEAKLY SEMIPRIME LEFT IDEALS IN TERNARY SEMIGROUPS

We start with the following theorem that gives a relation between prime and weakly semiprime ideal in ternary semigroup. Our starting points are the following definitions:

Definition 4.1. A proper left ideal P of a ternary semigroup S is said to be prime if, for any $a,b,c \in S, [abc] \in P$ implies $a \in P$ or $b \in P$ or $c \in P$.

Definition 4.2. A proper left ideal P of a ternary semigroup S is said to be prime if, for any $a \in S, [aaa] \in P$ implies $a \in P$.

Theorem 4.3. Let S be a ternary semigroup and A an ideal of S.

(i) If A is weakly prime and $S-A\neq\varnothing$, then $S-A$ is an m-system.

(ii) If $S-A$ is an m-system, then A is weakly prime.

Proof. (i) Assume that A is weakly prime and $S-A\neq\varnothing$. Let $a,b\in S-A$. To prove that there exists $c\in S-A$ such that $c\in[aSb]$ we suppose not. Then $c\notin[aSb]$, for every $c\in S-A$. Then $[aSb]\subseteq A$. Since A is weakly prime, we get $a\in A$ or $b\in A$. A contradiction. Hence $S-A$ is an m-system.

(ii) Assume that $S-A$ is an m-system. Let $a,b\in S$ be such that $[aSb]\subseteq A$. To show that $a\in A$ or $b\in A$, suppose not. Then $a\notin A$ and $b\notin A$. By assumption, there exists $c\in S-A$ such that $c\in[aSb]$. Let $c=[axb]$ for some $x\in S$. Since $[axb]\in A$, we have $c\in A$. A contradiction.

Theorem 4.4. Let S be a ternary semigroup and A an ideal of S.

(i) If A is weakly semiprime and $S-A\neq\varnothing$, then $S-A$ is an n-system.

(ii) If $S-A$ is an m-system, then A is weakly semiprime.

Proof. (i) Assume that A is weakly semiprime and $S-A\neq\varnothing$. Let $a\in S-A$. Suppose that $c\in[aSa]$ for every $c\in S\text{-}A$. Then $[aSa]\subseteq A$. This implies $a\in A$. A contradiction.

(ii) Assume that $S-A$ is an n-system. Let $a\in S$ be such that $[aSa]\subseteq A$. Suppose that $a\in S\text{-}A$. By assumption, there exists $c\in S-A$ such that $c\in[aSa]$. Let $c=[axa]$, for some $x\in S$. Since $[axa]\in A$, we obtain $c\in A$. A contradiction.

Theorem 4.5. Let S be a ternary semigroup, and let A be a weakly prime left ideal of S. Then $(A:a:b)_L$ is a weakly prime left ideal in S.

Proof. Assume that A is a weakly prime left ideal of S. By Lemma 3.1, we have $(A:a:b)_L$ is a left ideal in S. Let $[xyz]\in(A:a:b)_L$ and $x,y\notin(A:a:b)_L$. Since $[xyz]\in(A:a:b)_L$, we have

$$[xy[zab]]=[[xyz]ab]\in A$$

so that $[zab]\in A$. Therefore $z\in(A:a:b)_L$ and hence $(A:a:b)_L$ is a weakly prime left ideal in S.

Corollary 4.6. Let S be a ternary semigroup, and let A be a weakly prime left ideal of S. Then $(A:B:C)_L$ is a weakly prime left ideal in S.

Proof. Assume that A is a weakly prime left ideal of S. By Corollary 3.2, we have $(A:B:C)_L$ is a left ideal in S. Let $[xyz]\in(A:B:C)_L$ and $x,y\notin(A:B:C)_L$. Since $[xyz]\in(A:B:C)_L$, we have

$$[xy[zAB]]=[[xyz]BC]\subseteq A$$

so that $[zAB]\subseteq A$. Therefore $z\in(A:B:C)_L$ and hence $(A:B:C)_L$ is a weakly prime left ideal in S.

Theorem 4.7. Let S be a ternary semigroup, and let A be a weakly prime left ideal of S . Then $(A:a:b)_L$ is a weakly prime left ideal in S.

Proof. Assume that A is a weakly prime left ideal of S. By Lemma 3.1, we have $(A:a:b)_L$ is a left ideal in S. Let $[xSz\,]\in(A:a:b)_L$. Then

$$[xS[zab]]=[[xSz]ab]\subseteq A$$

so that $x\in A$ or $[zab]\in A$. Therefore $x\in(A:a:b)_L$ or $z\in(A:a:b)_L$ and hence $(A:a:b)_L$ is a weakly prime left ideal in S.

Corollary 4.8. Let S be a ternary semigroup, and let A be a weakly prime left ideal of S. Then $(A:B:C)_L$ is a weakly prime left ideal in S.

Proof. Assume that A is a weakly prime left ideal of S. By Corollary 3.2, we have $(A:B:C)_L$ is a left ideal in S. Let $[xSz]\subseteq (A:B:C)_L$. Then

$$[xS[zAB]]=[[xSz]BC]\subseteq A$$

so that $x\in A$ or $[zAB]\subseteq A$. Therefore $x\in (A:B:C)_L$ or $z\in (A:B:C)_L$ and hence $(A:B:C)_L$ is a weakly prime left ideal in S.

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