Some Basic Properties of Γ-Q-Algebra

Pairote Yiarayong[1] and Phakakorn Panpho[2]

[1] Department of Mathematics, Faculty of Science and Technology,
Pibulsongkram Rajabhat University,Phitsanuloke 65000, Thailand
[2] Faculty of Science and Technology, Pibulsongkram Rajabhat University,
Phitsanuloke 65000, Thailand

Corresponding author's email: pairote0027 [AT] hotmail.com

ABSTRACT— ***The purpose of this paper is to introduce the notion of a Γ-Q-algebras, we study Γ-ideals, Γ-subalgebras and upper sets in Γ-Q-algebras. Some characterizations of Γ-ideals and Γ-subalgebras are obtained. Moreover, we investigate relationships between Γ-ideals, Γ-subalgebras and upper sets in Γ-Q-algebras.***

Keywords— Γ-Q-algebra, Γ-ideal, Γ-subalgebra, upper set, Q-algebra.

1. INTRODUCTION

A Q-algebra is a nonempty set X with a constant 0 and a binary operation "$*$" satisfying axioms: for each $x, y, z \in X$,

1. $x * x = 0$,
2. $x * 0 = x$,
3. $(x * y) * z = (x * z) * y$

Y. Imai and K. Is´eki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras([5, 6]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [3, 5], Q. P. Hu and X. Li introduced a wide class of abstracts: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [8], J. Neggers, etc introduced the notion of an Q-algebra as a dualization of a generation of a BCK/BCI-algebras. In [1], Ahn and Kim introduced the notion of QS-algebra which is a generalization of Q-algebras. It is easy to see that every Q-algebras is Γ-Q-algebras.

In this paper is to introduce the notion of a Γ-Q-algebras, we study Γ-ideals, Γ-subalgebras and upper sets in Γ-Q-algebras. Some characterizations of Γ-ideals and Γ-subalgebras are obtained. Moreover, we investigate relationships between Γ-ideals, Γ-subalgebras and upper sets in Γ-Q-algebras.

2. BASIC PROPERTIES

In this section is to introduce the notion of a Γ-Q-algebras.

Definition 2.1. Let X and Γ be any nonempty sets. The structure $(\Gamma, X; 0)$ is called a Γ-Q-algebra If there exists a mapping $X \times \Gamma \times X \to X$ written as (x, γ, y) by $x\gamma y$, that satisfies the following condition

1. $x\gamma x = 0$, for all $x \in X$ and $\gamma \in \Gamma$,
2. $x\gamma 0 = x$, for all $x \in X$ and $\gamma \in \Gamma$,
3. $(x\gamma y)\beta z = (x\gamma z)\beta y$ for all $x, y, z \in X$ and. $\gamma, \alpha \in \Gamma$.

Throughout this paper X will denote a Γ-Q-algebra. We introduce a relation $\leq$ on X by $x \leq y$ if and only if $x\gamma y = 0$.

Example 2.2. Let $(\Gamma, X, 0)$ be an arbitrary Q-algebra and Γ any nonempty set. Define a mapping $X \times \Gamma \times X \to X$, by $x\gamma y \mapsto x * y$ for all $x, y \in X$ and $\gamma \in \Gamma$. It is easy to see that X is a Γ-Q-algebra. Indeed,

1. $x\gamma x = x * x = 0,$
2. $x\gamma 0 = x * 0 = x,$
3. $(x\gamma y)\beta z = (x * y)\beta z = (x * y) * z = (x * z) * y = (x\gamma z)\beta y$

$x, y, z \in X$ and. $\gamma, \alpha \in \Gamma$. Thus every Q-algebra implies a Γ-Q-algebra.

Example 2.3. Let $X = \{0, a, b, c, d\}$ in which "$\cdot$" is defined by

$\cdot$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	b	d
b	0	a	0	a	d
c	0	0	0	0	d
d	d	d	d	d	0

Let $\Gamma \neq \emptyset$. Define a mapping $X \times \Gamma \times X \to X$ by $x\gamma y = yx$ for all $x, y \in X$ and $\gamma \in \Gamma$. Then X is a Γ-Q-algebra. But it is not a Q-algebra because $d \cdot 0 = d \neq 0$.

Lemma 2.4. Let X be a Γ-Q-algebra. Then $x\gamma y = x\beta y$ for any $x, y \in X$ and $\gamma, \beta \in \Gamma$.

Proof. Let $x, y \in X$ and $\gamma, \beta \in \Gamma$. Then

$$\begin{aligned} x\gamma y &= (x\beta 0)\gamma y \\ &= (x\beta y)\gamma 0 \\ &= x\beta y. \end{aligned}$$

Hence $x\gamma y = x\beta y$.

Theorem 2.5. If X is a Γ-Q-algebra, then $(x\gamma(x\beta y))\alpha y = 0$, for any $x, y \in X$ and $\gamma, \alpha, \beta \in \Gamma$.

Proof. Let $x, y \in X$ and $\gamma, \alpha, \beta \in \Gamma$. Then

$$\begin{aligned} (x\gamma(x\beta y))\alpha y &= (x\gamma y)\alpha(x\beta y) \\ &= (x\gamma y)\alpha(x\gamma y) \\ &= 0. \end{aligned}$$

Hence $(x\gamma(x\beta y))\alpha y = 0$.

Theorem 2.6. If X is a Γ-Q-algebra, then $0\gamma(x\alpha y) = (0\gamma x)\alpha(0\gamma y)$ for any $x, y \in X$ and $\gamma, \alpha \in \Gamma.$

Proof. Let $x, y \in X$ and $\gamma, \alpha, \beta \in \Gamma.$ Then

$$\begin{aligned} 0\gamma(x\alpha y) &= ((0\beta y)\delta(0\beta y))\gamma(x\alpha y) \\ &= ((0\beta y)\delta(x\alpha y))\gamma(0\beta y) \\ &= (((x\beta y)\gamma x)\delta(x\alpha y))\gamma(0\beta y) \\ &= (((x\beta y)\gamma(x\alpha y))\delta x)\gamma(0\beta y) \\ &= (0\delta x)\gamma(0\beta y) \\ &= (0\gamma x)\alpha(0\gamma y). \end{aligned}$$

Hence $0\gamma(x\alpha y) = (0\gamma x)\alpha(0\gamma y).$

3. Γ-Q-ALGEBRAS

The results of the following lemmas seem play an important role to study e medial Γ-Q-algebra; these facts will be used so frequently that normally we shall make no reference to this definition.

Definition 3.1. A Γ-Q-algebra X is said to be medial if it satisfies the following property:

$$(x\gamma y)\alpha(z\beta u) = (x\gamma z)\alpha(y\beta u),$$

for any $x, y, z, u \in X$ and $\gamma, \alpha, \beta \in \Gamma.$

Lemma 3.2. If X is a medial Γ-Q-algebra, then $y\gamma x = 0\alpha(x\gamma y)$ for any $x, y \in X$ and $\gamma, \alpha \in \Gamma.$

Proof. Let $x, y \in X$ and $\gamma, \alpha, \beta \in \Gamma.$ Then

$$\begin{aligned} y\gamma x &= (y\gamma x)\alpha 0 \\ &= (y\gamma x)\alpha(y\gamma y) \\ &= (y\gamma y)\alpha(x\gamma y) \\ &= 0\alpha(x\gamma y). \end{aligned}$$

Hence $y\gamma x = 0\alpha(x\gamma y).$

Lemma 3.3. If X is a medial Γ-Q-algebra, then $x\gamma(y\alpha z) = z\gamma(y\alpha x)$ for any $x, y, z \in X$ and $\gamma, \alpha \in \Gamma.$

Proof. Let $x, y, z \in X$ and $\gamma, \alpha, \beta \in \Gamma.$ Then

$$\begin{aligned} x\gamma(y\alpha z) &= 0\beta((y\alpha z)\gamma x) \\ &= 0\beta((y\alpha x)\gamma z) \\ &= z\gamma(y\alpha x). \end{aligned}$$

Hence $x\gamma(y\alpha z) = z\gamma(y\alpha x).$

Lemma 3.4. If X is a medial Γ-Q-algebra, then $x\gamma(x\alpha y) = y$ for any $x, y \in X$ and $\gamma, \alpha \in \Gamma.$

Proof. Let $x, y \in X$ and $\gamma, \alpha, \beta \in \Gamma.$ Then

$$\begin{aligned} x\gamma(x\alpha y) &= 0\beta((x\alpha y)\gamma x) \\ &= 0\beta((x\alpha x)\gamma y) \\ &= y\gamma(x\alpha x) \\ &= y\gamma 0 \\ &= y\gamma 0. \end{aligned}$$

Hence $x\gamma(x\alpha y) = y.$

Lemma 3.5. If X is a medial Γ-Q-algebra, then $0\gamma(0\alpha y) = y$ for any $y \in X$ and $\gamma, \alpha \in \Gamma.$

Proof. Let $y \in X$ and $\gamma, \alpha \in \Gamma.$ Then

$$\begin{aligned} 0\gamma(0\alpha y) &= y\alpha 0 \\ &= y. \end{aligned}$$

Hence $0\gamma(0\alpha y) = y.$

Theorem 3.6. A Γ-Q-algebra X is medial if and only if it satisfies one of the following conditions: for any $x, y, z \in X$ and $\gamma, \alpha, \beta \in \Gamma.$

1. $y\gamma x = 0\alpha(x\gamma y)$
2. $x\gamma(y\alpha z) = z\gamma(y\alpha x)$
3. $x\gamma(x\alpha y) = y$
4. $0\gamma(0\alpha y) = y.$

Proof: $\Rightarrow$ It is clear.

$\Leftarrow$. Let $x, y, z \in X$ and $\gamma, \alpha, \beta \in \Gamma.$ Then

$$\begin{aligned} (x\gamma y)\alpha(z\beta u) &= u\alpha(z\beta(x\gamma y)) \\ &= u\alpha(y\beta(x\gamma z)) \\ &= (x\gamma z)\alpha(y\beta u) \end{aligned}$$

Hence $(x\gamma y)\alpha(z\beta u) = (x\gamma z)\alpha(y\beta u),$ which completes the proof.

4. Γ-IDEALS IN Γ-Q-ALGEBRAS

In this section we first introduce the notion of Γ-ideal in Γ-Q-algebras and next study some of elementary properties.

Definition 4.1. Let X be a Γ-Q-algebra and A a nonempty subset of X. A is said to be a Γ-ideal of X if it satisfies: $x \in X$ and $\gamma \in \Gamma$

1. $0 \in A$
2. $x\gamma y \in A$ and $y \in A$ imply that $x \in A$.

Obviously, $\{0\}$ and X are Γ-ideals of $X.$ We call $\{0\}$ and X the zero ideal and the trivial ideal of $X,$ respectively. An Γ-ideal A is said to be proper if $A \neq X.$

Definition 4.2. Let X be a Γ-Q-algebra and A a nonempty subset of X. A is said to be a Γ-subalgebra of X if $AA \subseteq A$.

Lemma 4.3. Let X be a Γ-Q-algebra. If A is a Γ-ideal of X, $X\Gamma A \subseteq A$ and $A\Gamma X \subseteq A.$

Proof. Let X be a Γ-Q-algebra X and let A be a Γ-ideal of X. Let $x \in X.$ Then $x\gamma x = 0 \in A,$ for all $\gamma \in \Gamma$ so $A\Gamma X \subseteq X\Gamma X \subseteq A.$

Definition 4.4. Let X be a Γ-Q-algebra and $a \in X, \gamma \in \Gamma$. Define $A(\gamma a)$ by

$$A(\gamma a) = \{0\} \cup \{x \in X : x\gamma a = 0\}..$$

Then we call $A(\gamma a)$ the initial section of the element a.

Definition 4.5. Let X be a Γ-Q-algebra and $a,b \in X, \gamma, \beta \in \Gamma$. Define $A(\gamma a, \beta b)$ by

$$A(\gamma a, \beta b) = \{0\} \cup \{x \in X : (x\gamma a)\beta b = 0\}.$$

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We call $A(\gamma a, \beta b)$ an upper set of a and b.

Theorem 4.6. Let X be a Γ-Q-algebra, $x, y \in X$ and $\gamma, \beta \in \Gamma$. Then

(1) $0 \in A(\gamma a)$, $0 \in A(\gamma a, \beta b)$.

(2) $a \in A(\gamma a)$.

(3) If $0\alpha y = 0$, then $A(\gamma a) \subseteq A(\gamma a, \beta b)$.

(4) If $0\alpha y \neq 0$, then $A(\gamma a) - \{0\} \subseteq X - A(\gamma a, \beta b)$.

(5) $A(\gamma a, \beta b) = A(\beta b, \gamma a)$.

Proof. Let X be a Γ-Q-algebra X, $a,b \in X$ and let $\gamma, \beta \in \Gamma$.

(1) By Definition 4.4 and Definition 4.5, we have $0 \in A(\gamma a)$ and $0 \in A(\gamma a, \beta b)$.

(2) Since $a\gamma a = 0$, we have $a \in A(\gamma a)$.

(3) Let $x \in A(\gamma a)$. Then $x \in \{x \in X : x\gamma a = 0\}$. or $x \in \{0\}$.

Case 1: $x \in \{0\}$.

Since $x \in \{0\}$, we get $x = 0$. It is easy to see that $x = 0 \in A(\gamma a, \beta b)$.

Case 2:

Since $x \in \{x \in X : x\gamma a = 0\}$,. it is easy to see that $x\gamma a = 0$. Then

$$\begin{aligned}(x\gamma a)\beta b &= 0\beta b \\ &= 0\alpha b \\ &= 0.\end{aligned}$$

Hence $A(\gamma a) \subseteq A(\gamma a, \beta b)$.

(4) Suppose that $0\alpha y \neq 0$. Let $x \in A(\gamma a) - \{0\}$. Then $x\gamma a = 0$ and $x \neq 0$. We have

$$\begin{aligned}(x\gamma a)\beta b &= 0\beta b \\ &= 0\alpha b \\ &\neq 0.\end{aligned}$$

Thus $x \notin A(\gamma a, \beta b)$ and hence $A(\gamma a) - \{0\} \subseteq X - A(\gamma a, \beta b)$.

(5) Since

$$\begin{aligned}A(\gamma a, \beta b) &= \{0\} \cup \{x \in X : (x\gamma a)\beta b = 0\} \\ &= \{0\} \cup \{x \in X : (x\gamma b)\beta a = 0\} \\ &= A(\beta b, \gamma a)\end{aligned}$$

we get $A(\gamma a, \beta b) = A(\beta b, \gamma a)$.

Theorem 4.7. Let X be a Γ-Q-algebra, $x, y, z \in X$ and $\gamma \in \Gamma$. If $x\gamma y = x\gamma z$, then $0\gamma y = 0\gamma z$.

Proof. By Definition 2.1, we get

$$\begin{aligned}(x\gamma y)\beta x &= (x\gamma x)\beta y \\ &= 0\beta y \\ &= 0\gamma y\end{aligned}$$

and

$$\begin{aligned}(x\gamma z)\beta x &= (x\gamma x)\beta z \\ &= 0\beta z \\ &= 0\gamma z.\end{aligned}$$

Since $(x\gamma y)\beta x = (x\gamma z)\beta x$, we have $0\gamma y = 0\gamma z$.

5. CONCLUSION

Many new classes of Γ-Q-algebras have been discovered recently. All these have attracted researchers of the field to investigate these newly discovered classes in detail. This article investigates we study medial Γ-Q-algebras.

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