Some Basic Properties of Γ-Abel-Grassmann's Groupoids

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ABSTRACT— ***In this article we investigate some basic properties of newly discovered classes of Γ-AG-groupoids. We consider three classes that include Γ-AG*-groupoids, Γ-middle nuclear square AG-groupoids, Γ-right nuclear square AG-groupoids and Γ-Bol*-AG-groupoids. We start with the following theorem that gives a relation between Γ-AG*-groupoids, Γ-middle nuclear square and Γ- nuclear square. We investigate that every Γ-cancellative AG*-groupoid is Γ-transitively commutative AG-groupoid.***

Keywords— : Γ-AG-groupoid, Γ-AG*-groupoid, Γ-AG**-groupoid, Γ-cancellative AG*-groupoid, Γ-Bol*-AG-groupoid.

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1. INTRODUCTION

Abel-Grassmann's groupoid (AG-groupoid) is the generalization of semigroup theory with the wide range of usages in theory of flocks [7]. The fundamentals of this non-associative algebraic structure were the first discovered by Kazim and Naseeruddin [1]. AG-groupoid is a non-associative algebraic structure mid way between a groupoid and a commutative semigroup. It is interesting to note that an AG-groupoid with right identity becomes a commutative monoid [6]. This structure is closely related with a commutative semigroup because if an AG-groupoid contains a right identity, then it becomes a commutative monoid [6]. A left identity in an AG-groupoid is unique. Ideals in AG-groupoids have been discussed by Mushtaq and Yousuf [5, 6].

In 1981, the notion of Γ-semigroups was introduced by Sen. Let S and Γ be any nonempty sets. If there exists a mapping $S\times\Gamma\times S\rightarrow S$ written (a,α,c) by $a\alpha c$, S is called a Γ-semigroup if S satisfies the identity:

$$(a\alpha b)\beta c = a\alpha(b\beta c)$$

for all $a,b,c\in S$ and $\alpha,\beta\in\Gamma$. A Γ-AG-groupoid S is called Γ-AG-band if $a\gamma a = a$ for all $a\in S,\gamma\in\Gamma$. In this paper we are going to investigate some interesting properties of newly discovered classes of namely; Γ-AG-groupoid S always satisfies the Γ-medial law:

$$(a\gamma b)\beta(c\delta d) = (a\gamma c)\beta(b\delta d)$$

for all $a,b,c,d\in S$ and $\gamma,\beta,\delta\in\Gamma$ (See [2]), while a Γ-AG-groupoid S with left identity e always satisfies Γ-paramedial law:

$$(a\gamma b)\beta(d\delta b) = (c\gamma a)\beta(b\delta d)$$

for all $a,b,c,d\in S,\gamma,\beta,\delta\in\Gamma$ (See[2]).

Now we define the concepts that we will use. A Γ-AG-groupoid S is called Γ-transitively commutative if $a\gamma b = b\gamma a$ and $b\gamma c = c\gamma b$ for all $a,b,c\in S,\gamma\in\Gamma$ implies $a\gamma c = c\gamma a$. A Γ-AG-groupoid S is called $\Gamma - T^1$-AG-groupoid if $a\gamma b = c\gamma d$ for all $a,b,c,d\in S,\gamma\in\Gamma$ implies $b\gamma a = d\gamma c$. A Γ-AG-groupoid S is called Γ-left nuclear square if $a^2\gamma(b\delta c) = (a^2\gamma b)\delta c$ for all $a,b,c,d\in S,\gamma,\delta\in\Gamma$. Γ-Right nuclear and nuclear square can be defined analogously. A Γ-AG-groupoid S is called Γ-Bol*-groupoid if it satisfies the identity $a\gamma((b\delta c)\beta d)) = ((a\gamma b)\delta c))\beta d$ for all $a,b,c,d \in S, \gamma,\delta,\beta \in \Gamma$. A groupoid S is called Γ-left cancellative if $a\gamma b = a\gamma c$ for all $a,b,c,d \in S, \gamma \in \Gamma$ implies $b = c$. Γ-right cancellative and Γ-cancellative AG-groupoid can be defined similarly.

Furthermore, in this paper we investigate elementary properties of a commutative Γ-AG-groupoids. We, also indicated the non similarity of a Γ-AG-groupoids to the usual notion of a Γ-AG*-groupoids.

2. BASIC RESULTS

In this section we refer to [13, 14, 16] for some elementary aspects and quote few definitions and examples which are essential to step up this study. For more detail we refer to the papers in the references.

Example 2.1. [14] (1). Let S be an arbitrary AG-groupoid and Γ any non-empty set. Define a mapping $S \times \Gamma \times S \to S$; by $a\gamma b = ab$ for all $a,b \in S, \gamma \in \Gamma$. It is easy to see that S is a Γ-AG-groupoid.
(2). Let $\Gamma = \{1,2,3\}$. Define a mapping $\mathbb{Z} \times \Gamma \times \mathbb{Z} \to \mathbb{Z}$ by $a\gamma b = b - \gamma - a$ for all $a,b \in \mathbb{Z}, \gamma \in \Gamma$ where "-" is a usual subtraction of integers. Then $\mathbb{Z}$ is a Γ-AG-groupoid.

Lemma 2.2. [16] Every Γ-AG-groupoid is Γ-medial.

Lemma 2.3 [13, 16] Let S be a Γ-AG-groupoid with a left identity; then $a\gamma(b\alpha c) = b\gamma(a\alpha c)$ for all $a,b,c \in S, \gamma,\alpha \in \Gamma$.

Lemma 2.4. [16] If a is an arbitrary element of a locally associative Γ-AG-groupoid with a left identity, then for every $\gamma,\alpha \in \Gamma$ and every positive integer n, $a_\alpha^n = a_\beta^n$.

3. Γ-AG-GROUPOIDS

We start with the following theorem that gives a relation between Γ-right nuclear square, Γ-middle nuclear square and Γ- nuclear square. Our starting point is the following lemma:

Lemma 3.1. If S is a Γ-AG-groupoid with left identity, then $a\gamma b = a\beta b$ for all $a,b \in S$ and $\gamma,\beta \in \Gamma$.

Proof. Let S be a Γ-AG-groupoid and e be the left identity of S, $a,b \in S$ and let $\gamma,\beta \in \Gamma$ therefore we have

$$\begin{aligned} a\gamma b &= a\gamma(e\beta b) \\ &= e\gamma(a\beta b) \\ &= a\beta b. \end{aligned}$$

Hence $a\gamma b = a\beta b$.

Lemma 3.2. Let S be a Γ-AG-groupoid. Then the following are equivalent:

i. $a\gamma b = c\gamma d \Rightarrow a\gamma c = b\gamma d, \forall a,b,c,d \in S, \gamma \in \Gamma.$

ii. $a\gamma b = c\gamma d \Rightarrow c\gamma a = d\gamma b, \forall a,b,c,d \in S, \gamma \in \Gamma$.

Proof. i.$\Rightarrow$ ii. Let S be a Γ-AG-groupoid, $a,b,c,d \in S$ and let $\gamma \in \Gamma$ such that $a\gamma b = c\gamma d$. Then

$$\begin{aligned} c\gamma d &= a\gamma b \\ c\gamma a &= d\gamma b. \end{aligned}$$

Hence $c\gamma a = d\gamma b$.

ii. $\Rightarrow$ i. The proof is obvious.

Proposition 3.3. Let S be a Γ-AG-groupoid. Then S is a Γ-commutative semigroup if

$$a\gamma b = c\gamma d \Rightarrow a\gamma d = b\gamma c \text{ for all } a,b,c,d \in S, \gamma \in \Gamma \text{..........(*)}$$

Proof. Since $\forall a,b \in S$ and $\forall \gamma \in \Gamma$ the equation $a\gamma b = a\gamma b$ trivially holds. Now an application of (*) proves commutativity in S. Since any commutative Γ-AG-groupoid S is associative, thus S becomes commutative Γ-semigroup.

Corollary 3.4. Let S be a Γ-AG-groupoid. Then S isa Γ-commutative semigroup if

$$a\gamma b = c\gamma d \Rightarrow d\gamma a = c\gamma b \text{ for all } a,b,c,d \in S \text{ and } \gamma \in \Gamma.$$

Proof. Follows from Proposition 3.3.

Theorem 3.5. Let S be a Γ-right alternative AG-groupoid; then $(a^2_\alpha \gamma b)\beta c = a^2_\beta \alpha(c\gamma b)$ for all $a,b,c \in S$ and $\gamma, \alpha, \beta \in \Gamma$.

Proof. Let S be a Γ-AG-groupoid, $a,b,c \in S$ and let $\gamma, \alpha, \beta \in \Gamma$. Then

$$\begin{aligned}
(a^2_\alpha \gamma b)\beta c &= ((a\alpha a)\gamma b)\beta c \\
&= (c\gamma b)\beta(a\alpha a) \\
&= ((c\gamma b)\beta a)\alpha a \\
&= ((c\gamma b)\beta a)\alpha a \\
&= ((a\gamma b)\beta c)\alpha a \\
&= (a\beta c)\alpha(a\gamma b) \\
&= (a\beta a)\alpha(c\gamma b) \\
&= a^2_\beta \alpha(c\gamma b).
\end{aligned}$$

Hence $(a^2_\alpha \gamma b)\beta c = a^2_\beta \alpha(c\gamma b)$.

Corollary 3.6. Let S be an a Γ-AG-groupoid with left identity. If S is a Γ-right alternative AG-groupoid, then $(a^2\gamma b)\beta c = a^2\gamma(c\beta b)$ for all $a,b,c \in S$ and $\gamma, \beta \in \Gamma$.

Proof. The proof is obvious.

Theorem 3.7. Let S bean a Γ-AG-groupoid with left identity. If S is a Γ-right alternative AG-groupoid, then S is a Γ-left nuclear square.

Proof. Let S be a Γ-AG-groupoid, $a,b,c \in S$ and let $\gamma, \beta, \alpha \in \Gamma$. Then

$$\begin{aligned}
(a^2\gamma b)\beta c &= ((a\alpha a)\gamma b)\beta c \\
&= (c\gamma b)\beta(a\alpha a) \\
&= ((c\gamma b)\beta a)\alpha a \\
&= ((a\gamma b)\beta c)\alpha a \\
&= (a\beta c)\alpha(a\gamma b) \\
&= (a\beta a)\alpha(c\gamma b) \\
&= a^2\alpha(c\gamma b) \\
&= a^2\gamma(c\beta b).
\end{aligned}$$

Hence S is a Γ-left nuclear square.

Theorem 3.8. In a locally associative Γ-AG-groupoid S with left identity and let S be a Γ-right alternative AG-groupoid. If S is a Γ-right nuclear square, then S is a Γ-middle nuclear square.

Proof. Suppose S is a Γ-right nuclear square. Let $a,b,c \in S$ and $\gamma,\beta,\alpha \in \Gamma$. Then

$$\begin{aligned}(a\gamma b^2)\alpha c &= (c\gamma b^2)\alpha a \\ &= (c\gamma(b\beta b))\alpha a \\ &= ((c\gamma b)\beta b)\alpha a \\ &= ((b\gamma b)\beta c)\alpha a \\ &= (a\beta c)\alpha(b\gamma b) \\ &= a\beta(c\alpha(b\gamma b)) \\ &= a\beta((c\alpha b)\gamma b) \\ &= a\beta((b\alpha b)\gamma c) \\ &= a\beta(b^2\gamma c) \\ &= a\gamma(b^2\alpha c).\end{aligned}$$

Thus S is a Γ-middle nuclear square.

Theorem 3.9. In a locally associative Γ-AG-groupoid S with left identity and let S be a Γ-right alternative AG-groupoid. If S is a Γ- nuclear square, then S is a Γ-right nuclear square

Proof. The proof is obvious.

Theorem 3.10. In a locally associative Γ-AG-groupoid S with left identity and let S be a Γ-right alternative AG-groupoid. If S is a Γ-middle nuclear square, then S is a Γ-nuclear square.

Proof. Suppose S is a Γ-middle nuclear square. Let $a,b,c \in S$ and let $\gamma,\alpha,\beta \in \Gamma$. Then

$$\begin{aligned}(a\gamma b)\alpha c^2 &= (c^2\gamma b)\alpha a \\ &= ((c\beta c)\gamma b)\alpha a \\ &= ((b\beta c)\gamma c)\alpha a \\ &= (b\beta(c\gamma c))\alpha a \\ &= (a\beta(c\gamma c))\alpha b \\ &= a\beta((c\gamma c)\alpha b) \\ &= a\beta((b\gamma c)\alpha c) \\ &= a\beta(b\gamma(c\alpha c)) \\ &= a\beta(b\gamma c^2) \\ &= a\gamma(b\alpha c^2).\end{aligned}$$

Thus S is also Γ-middle nuclear square. But by Theorem 3.7, we have S is a Γ-left nuclear square as well. Hence S is a Γ-right nuclear square.

Proposition 3.11. Every Γ anti-commutative AG-groupoid S is Γ-transitively commutative.

Proof. Let S be a Γ-anti-commutative AG-groupoid, $a,b,c \in S$ and let $\gamma \in \Gamma$ such that

$a\gamma b = b\gamma a, b\gamma c = c\gamma b.$

Then by definition of Γ-anti-commutativity, this implies that $a = b, b = c.$ But this implies that $a = c$ and which further implies that $a\gamma c = c\gamma a$. Hence S is a Γ-transitively commutative.

Remark 3.12. Every Γ-anti-commutative AG-groupoid S is Γ-cancellative but the converse is not true.

Example 3.13. A Γ-transitively commutative AG-groupoid of order 4.

A. ·	B. 1	C. 2	D. 3	E. 4
F. 1	G. 1	H. 1	I. 1	J. 1
K. 2	L. 1	M. 1	N. 1	O. 1
P. 3	Q. 1	R. 1	S. 1	T. 1
U. 4	V. 2	W. 2	X. 2	Y. 1

Define a mapping $S\times\Gamma\times S\rightarrow S$ by $a\gamma b=a\cdot b$; for all $a,b\in S$ and $\gamma\in\Gamma$. Then S is Γ-transitively commutative. Also, we can see that S is not a Γ anti-commutative AG-groupoid.

Theorem 3.14. Let S be a Γ-anti-commutative AG-groupoid. Then the following are equivalent.

i. S is Γ-AG-band;

ii. S is Γ-locally associative.

Proof. i.$\Rightarrow$ii.is always true ii.

ii.$\Rightarrow$i. By definition of Γ-locally associativity and Γ-anti-commutativity, for every $a\in S$ and $\gamma\in\Gamma$, we have $a\gamma a^2=a^2\gamma a\Rightarrow a^2=a$.

4. Γ-AG*-GROUPOIDS

We start with the following theorem that gives a relation between Γ-AG*-groupoids, Γ-middle nuclear square AG-groupoids, Γ-right nuclear square AG-groupoids and Γ-Bol*-AG-groupoids. Our starting point is the following lemma:

Lemma 4.1. Let S be a Γ-AG-groupoidswith left identity. If S is a Γ-AG*-groupoid, then S is a Γ-Bol*-AG-groupoid.

Proof. Let S be a Γ-AG*-groupoid, $a,b,c,d\in S$ and let $\gamma,\alpha\in\Gamma$. Then by definition of Γ-AG*-groupoid $(a\gamma b)\alpha c=b\gamma(a\alpha c)$. Now since

$$\begin{aligned}((a\gamma b)\alpha c)\beta d &= (d\alpha c)\beta(a\gamma b)\\ &= (d\alpha a)\beta(c\gamma b)\\ &= a\alpha(d\beta(c\gamma b))\\ &= a\alpha((c\beta d)\gamma b)\\ &= a\alpha((b\beta d)\gamma c)\\ &= a\alpha(d\beta(b\gamma c))\\ &= (d\alpha a)\beta(b\gamma c)\\ &= ((b\gamma c)\alpha a)\beta d\\ &= a\alpha((b\gamma c)\beta d)\\ &= a\gamma((b\alpha c)\beta d)\end{aligned}$$

we have S is commutative -Γ-Bol*-AG-groupoid.

Theorem 4.2. In a locally associative Γ-AG-groupoid S with left identity and let S be a Γ-AG*-groupoid. Then

i. S is Γ-middle nuclear square AG-groupoid;

ii. S is Γ-right nuclear square AG-groupoid.

Proof. i. Let $a,b,c \in S$ and let $\gamma,\alpha,\beta \in \Gamma.$ Then

$$\begin{aligned}(a\gamma b^2)\alpha c &= b^2\gamma(a\alpha c)\\ &= (b\beta b)\gamma(a\alpha c)\\ &= (b\beta a)\gamma(b\alpha c)\\ &= a\beta(b\gamma(b\alpha c))\\ &= a\beta((b\gamma b)\alpha c))\\ &= a\beta(b^2\alpha c)\\ &= a\gamma(b^2\alpha c).\end{aligned}$$

Hence S is a Γ-middle nuclear square AG-groupoid.

ii. Let $a,b,c \in S$ and let $\gamma,\alpha,\beta \in \Gamma.$ Then

$$\begin{aligned}a\gamma(b\alpha c^2) &= (b\gamma a)\alpha c^2\\ &= (b\gamma c)\alpha(a\beta c)\\ &= ((a\beta c)\gamma c)\alpha b\\ &= ((c\beta c)\gamma a)\alpha b\\ &= a\gamma((c\beta c)\alpha b)\\ &= a\gamma(c\beta(c\alpha b))\\ &= (c\gamma a)\beta(c\alpha b)\\ &= (c\gamma c)\beta(a\alpha b)\\ &= ((a\alpha b)\gamma c)\beta c\\ &= (b\alpha(a\gamma c))\beta c\\ &= (a\gamma c)\alpha(b\beta c)\\ &= (a\gamma b)\alpha c^2.\end{aligned}$$

Hence S is a Γ-right nuclear square AG-groupoid.

Theorem 4.3. Every Γ-right cancellative AG*-groupoid is Γ-transitively commutative AG-groupoid.

Proof. Let S be a Γ-right cancellative AG*-groupoid, $a,b,c,d \in S$ and let $\gamma,\alpha \in \Gamma$ such that $a\gamma b = b\gamma a,$ $b\gamma c = c\gamma b.$ Then

$$\begin{aligned}(a\gamma c)\gamma b &= c\gamma(a\gamma b)\\ &= c\gamma(a\gamma b)\\ &= c\gamma(b\gamma a)\\ &= (b\gamma c)\gamma a\\ &= (c\gamma b)\gamma a\\ &= (a\gamma b)\gamma c\\ &= (b\gamma a)\gamma c\\ &= (c\gamma a)\gamma b\\ a\gamma c &= c\gamma a.\end{aligned}$$

Hence S is Γ-transitively commutative AG-groupoid.

Proposition 4.4. Every Γ-AG*-groupoid is Γ-left alternative AG-groupoid.

Proof. Let S be a Γ-AG*-groupoid, $a,b \in S$ and let $\gamma,\alpha \in \Gamma$. Then $(a\gamma a)\alpha b = a\gamma(a\alpha b)$. Hence S is a Γ-left alternative AG-groupoid.

Theorem 4.5. Let S be a Γ-AG-groupoids with left identity. If S is a Γ-right alternative AG**-groupoid, then S is a Γ-nuclear square.

Proof. Let S be a Γ-right alternative AG**-groupoid, $a,b,c \in S$ and let $\gamma,\alpha,\beta \in \Gamma$. Then

$$\begin{aligned} a\gamma(b\alpha c^2) &= b\gamma(a\alpha c^2) \\ &= b\gamma(a\alpha(c\beta c)) \\ &= b\gamma((a\alpha c)\beta c) \\ &= (a\alpha c)\gamma(b\beta c) \\ &= (a\alpha b)\gamma(c\beta c) \\ &= (a\alpha b)\gamma c^2. \\ &= (a\gamma b)\alpha c^2. \end{aligned}$$

Hence S is a Γ-nuclear square.

Proposition 4.6. Every $\Gamma - T^1$-AG-groupoid S is AG**-groupoid.

Proof. Let S be a $\Gamma - T^1$-AG-groupoid, $a,b,c \in S$ and let $\gamma,\alpha \in \Gamma$ .Then

$$\begin{aligned} (a\gamma b)\alpha c &= (c\gamma b)\alpha a \\ c\alpha(a\gamma b) &= a\alpha(c\gamma b). \end{aligned}$$

Hence S is Γ-AG**-groupoid.

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