Some Basic Properties of Weakly Completely Primary Ideals in Γ-Near Rings

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ABSTRACT— *In this paper, we study completely primary and weakly completely primary ideals in Γ-near-rings. Some characterizations of completely primary and weakly completely primary ideals are obtained. Moreover, we investigate relationships completely primary and weakly completely primary ideals in Γ-near rings. Finally, we obtain necessary and sufficient conditions of a weakly completely primary ideal to be a completely primary ideals in Γ-near rings.*

Keywords— Γ-near-ring, completely primary, weakly completely primary , quasi completely weakly primary, quasi completely primary.

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1. INTRODUCTION

Throughout this paper, by a Γ-near-ring N we always mean a zero-symmetric near-ring with identity 1. For basic definitions in near-rings one may refer [20]. In 1970 W. L. M. Holcombe was introducing the notions of (0, 1, 2)-prime ideals of a near ring. In 1977 G. Pilz, was introducing the notion of prime ideals of a near ring. In 1988 N.J.Groenewald was introducing the notions of completely (semi) prime ideals of a near ring. In 1991 N.J.Groenewald was introducing the notions of 3-(semi) prime ideals of a near ring . In 2003 D. D. Anderson and E. Smith defined weakly prime ideals in commutative rings, an ideal P of a ring R is weakly prime if $0 \neq ab \in P$ implies $a \in P$ or $b \in P.$

The concept of Γ-near ring, a generalization of both the concepts near-ring and Γ-ring was introduced by Satyanarayana [21]. Later, several authors such as Booth and Booth, Groenewald [4, 5, 6] studied the ideal theory of Γ-near rings. Groenewald [12] introduced semi uniformly strongly prime near-rings.

In this paper we study completely primary and weakly completely primary ideals in Γ-near-rings. Some characterizations of completely primary and weakly completely primary ideals are obtained. Moreover, we investigate relationships completely primary and weakly completely primary ideals in Γ-near rings. Finally, we obtain necessary and sufficient conditions of a weakly completely primary ideal to be a completely primary ideals in Γ-near rings.

2. BASIC RESULTS

In this section we refer to [24, 25] for some elementary aspects and quote few theorem and lemmas which are essential to step up this study. For more details we refer to the papers in the references.

Definition 2.1. [25] All near-rings considered in this paper are left distributive. A Γ-near-ring is a triple $(N,\Gamma,+)$, where

(i) $(N,+)$ is a group (not necessarily abelian);

(ii) Γ is a non-empty set of binary operations on N such that for each $\gamma \in \Gamma$, $(N,+,\gamma)$ is a right near -ring and;

(iii) $(a\gamma b)\alpha c = a\gamma(b\alpha c)$, for all $a,b,c \in N$ and $\gamma,\alpha \in \Gamma.$

Γ-near rings generalize near-rings in the sense that every near-ring N is a Γ-near ring, with $\Gamma = \{\cdot\}$, where $\cdot$ is the multiplication defined on N.

Definition 2.2. [25] Let N be a Γ-near ring, then a normal subgroup A of $(N,\Gamma,+)$ is said to be

(i) left ideal if $m\gamma(n + a) - m\gamma n \in A$, for all $a \in A, \gamma \in \Gamma$, and $m,n \in N$;
(ii) right ideal if $a\gamma n \in A$, for all $a \in A, \gamma \in \Gamma$, and $n \in N$;
(iii) ideal if it is both left and right ideal of N.

If A is an ideal of N, then it is denoted by $A \lhd N$. The ideal generated by $a \in N$, is denoted by $< a >$.

Lemma 2.3. [25] Let A be a left ideal of a Γ-near ring N. Then $(A:n)_\gamma$ is a left ideal of N, where $(A:n)_\gamma = \{m \in N : m\gamma n \in A\}$.

Lemma 2.4. Let A be an ideal of $(N,+,\cdot)$. Then is a Γ-near-ring under the operations: For all $a,b \in N$

$$(a+A)+(b+N) = (a+b)+A \text{ and } (a+A)(b+N) = (ab)+A.$$

Lemma 2.5. Let A and B be ideals of $(N,\Gamma,+)$. Then $(A+B)/A \approx B/(A \cap B)$. Furthermore, if $A \subseteq B$, then $(N/A)/(B/A) \approx N/B$.

Definition 2.6. Let $(N,\Gamma,+)$ be a Γ-near-ring and A be a subset of N. We write

$$\sqrt{A} = \{a \in N : a^k \in A \text{ for some positive integer } k\}.$$

Definition 2.7. A ideal P of a Γ-near-ring N is called a completely primary ideal if for $a,b \in N$ and $\gamma \in \Gamma$ such that $a\gamma b \in P$ implies that $a^n \in P$ or $b \in P$, for some positive integer n.

Definition 2.8. A ideal P of a Γ-near-ring N is called a weakly completely primary ideal if for $a,b \in N$ and $\gamma \in \Gamma$ such that $0 \neq a\gamma b \in P$ implies that $a^n \in P$ or $b \in P$, for some positive integer n.

Clearly every completely primary ideal is weakly completely primary and $\{0\}$ is always weakly completely primary ideal of N. The following example shows that a weakly completely primary ideal need not be a completely primary ideal in general.

Example 2.9. Let $N = \{0, a, b, c, d, 1, 2, 3\}$ and $\Gamma = \{0,1\}$. Define addition and multiplication in N as follows:

$+$	0	1	2	3	a	b	c	d
0	0	1	2	3	a	b	c	d
1	1	2	3	0	d	c	a	b
2	2	3	0	1	b	a	d	c
3	3	0	1	2	c	d	b	a
a	a	d	b	c	2	0	1	3
b	b	c	a	d	0	2	3	1
c	c	a	d	b	1	3	0	2
d	d	b	c	a	3	1	2	0

$\cdot$	0	1	2	3	a	b	c	d
0	0	0	0	0	0	0	0	0
1	0	1	2	3	a	b	c	d
2	0	2	0	2	2	2	0	0
3	0	3	2	1	b	a	c	d
a	0	a	2	b	a	b	c	d
b	0	b	2	a	b	a	c	d
c	0	c	0	c	0	0	0	0
d	0	d	0	d	2	2	0	0

Then $(N,+,\cdot)$ is a Γ-near ring. Here $\{0,\ c\}$ is a weakly completely primary ideal, but not a completely primary, since $2\cdot\gamma\cdot 2 = 0 \in \{0,\ c\}$.

3. MAIN RESULTS

We start with the following theorem that gives a relation between weakly completely primary and completely primary ideals in a Γ-near-ring. Our starting points is the following lemma:

Lemma 3.1. If N is a Γ-near-ring with identity, then $a\gamma b = a\alpha b$ for all $a,b \in N$ and $\gamma,\alpha \in \Gamma$.

Proof. Let N be a Γ-near-ring and e be the identity of N, and let $a,b \in N, \gamma,\alpha \in \Gamma$ therefore we have

$$\begin{aligned} a\gamma b &= a\gamma(e\alpha b) \\ &= (a\gamma e)\alpha b \\ &= a\alpha b. \end{aligned}$$

Hence $a\gamma b = a\alpha b.$

Lemma 3.2. Let N be a Γ-near-ring, and let A be a left ideal of N. Then $(A:\Gamma:B)$ is a left ideal in N, where $(A:\Gamma:B) = \{n \in N : n\Gamma B \subseteq A\}$.

Proof. Let N be a Γ-near-ring, and let A be a left ideal of N. Suppose that $n \in N$ and $m,n \in (A:\Gamma:B)$. Then $m\Gamma B \subseteq A$ and $n\Gamma B \subseteq A$ so that

$$(n-m)\Gamma B = n\Gamma B - m\Gamma B \subseteq A.$$

Therefore $n-m \in (A:\Gamma:B)$. For $a \in (A:\Gamma:B)$ and $n \in N$,

$$\begin{aligned} (n+a-n)\Gamma B &= n\Gamma B + a\Gamma B - n\Gamma B \\ &\subseteq n\Gamma B + A - n\Gamma B \\ &\subseteq A \end{aligned}$$

since A is a left ideal of N. Therefore, $n+a-n \in (A:\Gamma:B)$. Thus $(A:\Gamma:B)$ is a normal subgroup of N. Let $m,n \in N, a \in (A:\Gamma:B)$ and β $\gamma \in \Gamma$. Then

$$\begin{aligned} (m\gamma(n-a) - m\gamma n)\Gamma B &= (m\gamma(n-a))\Gamma B - (m\gamma n)\Gamma B \\ &= m\gamma((n-a)\Gamma B) - (m\gamma n)\Gamma B \\ &= m\gamma(n\Gamma B - a\Gamma B) - (m\gamma n)\Gamma B \\ &= m\gamma(n\Gamma B - a\Gamma B) - (m\gamma n)\Gamma B \\ &\subseteq A. \end{aligned}$$

Thus $m\gamma(n-a) - m\gamma n \in (A:\Gamma:B)$. Hence $(A:\Gamma:B)$ is a left ideal in N.

Theorem 3.3. Let N be a Γ-near-ring, and let A be an ideal of N. If A is a weakly quasi completely primary (quasi completely primary) ideal of N, then $(A:\Gamma:B)$ is a weakly quasi completely primary (quasi completely primary) ideal in N, where $B \not\subset A$.

Proof. Let N be a Γ-near-ring, and let A be a weakly completely quasi primary ideal of N. Suppose that $0 \neq m\gamma n \in (A:\Gamma:B)$ and $m^k \notin (A:\Gamma:B)$, for all positive integer k. Then

$$0 \neq m\gamma(n\Gamma B) = (m\gamma n)\Gamma B \subseteq A.$$

By Definition of weakly quasi completely primary ideal, we get $m^k \in A$ or $n\Gamma B \subseteq A$ for some positive integer k so that $n \in (A:\Gamma:B)$. Hence $(A:\Gamma:B)$ is a weakly quasi completely primary ideal in N.

Corollary 3.4. Let N be a Γ-near-ring, and let A be a weakly quasi completely primary (quasi completely primary) ideal of N. Then $(A:m)_\gamma$ is a weekly quasi completely primary (quasi completely primary) ideal in N, where $m \in N - A$.

Proof. This follows from Theorem 3.3.

Theorem 3.5. Let N be a Γ-near-ring, and let P be an ideal of N. If P is a weakly completely primary ideal that is not completely primary. Then $\sqrt{P} = \sqrt{0}$.

Proof. Let N be a Γ-near-ring with identity. First, we prove that $P^2 = 0$. Suppose that $P^2 \neq 0$ we show that P is weakly completely primary. Let $a\gamma b \in P$, where $a, b \in N, \gamma \in \Gamma$. If $a\gamma b \neq 0$, then either

$$a \in \sqrt{P} \text{ or } b \in P$$

since P is weakly completely primary ideal. So suppose that $a\gamma b = 0$. If $P\gamma b \neq 0$, then there is an element p' of P such that $p'\gamma b \neq 0$, so that

$$0 \neq p'\gamma b = (p' + a)\gamma b \in P,$$

and hence P weakly completely primary ideal gives either $p' + a \in \sqrt{P}$ or $b \in P$. As $p' + a \in \sqrt{P}$ and $p' \in P \subseteq \sqrt{P}$ we have either $a \in \sqrt{P}$ or $b \in P$. So we can assume that $P\gamma b = 0$. Similarly, we can assume that $P\gamma a = 0$. Since $P^2 \neq 0$, there exist $c, d \in P$ such that $c\gamma d \neq 0$. Then

$$(a + c)\gamma(b + d) \in P,$$

so either $p + c \in P$ or $q + d \in \sqrt{P}$, and hence either $p \in P$ or $q \in \sqrt{P}$. Thus P is completely primary ideal.

Clearly, $\sqrt{0} \subseteq \sqrt{P}$. As $P^2 = 0$, we get $\sqrt{P} \subseteq \sqrt{0}$, hence $\sqrt{P} = \sqrt{0}$, as required.

Corollary 3.6. Let N be a Γ-near-ring, and let P an ideal of N. If $\sqrt{P} \neq \sqrt{0}$, then P is completely primary if and only if P is weakly completely primary.

Proof. This follows from Theorem 3.5.

Lemma 3.7. Let N be a Γ-near-ring with identity, and let P be a proper ideal of N. If P is a weakly completely primary ideal of N, then $(P:\Gamma:N\Gamma a) = P \cup (0:\Gamma:N\Gamma a)$, where $a \in N - \sqrt{P}$.

Proof. Let N be a Γ-near-ring with identity, and let P be a weakly completely primary ideal of N. Clearly,

$$P \cup (0:\Gamma:N\Gamma a) \subseteq (P:\Gamma:N\Gamma a).$$

For the other inclusion, suppose that $m \in (P:\Gamma:N\Gamma a)$, so that

$$m\Gamma(N\Gamma a) \subseteq P.$$

If $0 \neq m\Gamma(N\Gamma a)$, then $N\Gamma a \subseteq P$ since P is weakly completely primary. If $0 = m\Gamma(N\Gamma a)$, then $m \in (0:\Gamma:N\Gamma a)$ so we have the equality.

Corollary 3.8. Let N be a Γ-near-ring with identity, and let P be a proper ideal of N. If P is a weakly completely primary ideal of N, then $(P:\Gamma:a) = P \cup (0:\Gamma:a)$, where $a \in N - \sqrt{P}$.

Proof. This follows from Lemma 3.7.

Corollary 3.9. Let N be a Γ-near-ring with identity, and let P be a proper ideal of N. If $(P:\Gamma:N\Gamma a) = P \cup (0:\Gamma:N\Gamma a)$, then $(P:\Gamma:N\Gamma a) = P$ or $(P:\Gamma:N\Gamma a) = (0:\Gamma:N\Gamma a)$, where $a \in N - \sqrt{P}$.

Proof. This follows from Lemma 3.7.

Theorem 3.10. Let N be a Γ- near-ring with identity, and let P be a proper ideal of N. If $(P:\Gamma:n) = P$ or $(P:\Gamma:n) = (0:\Gamma:n)$, then P is a weakly completely primary ideal of N, where $n \in N - \sqrt{P}$.

Proof. Let N be a Γ-near-ring with identity, and let P be a proper ideal of N. Suppose that Let $0 \neq m\gamma n \in P$, where $m \in N - \sqrt{P}, \gamma \in \Gamma$. Then

$$m \in (P:\Gamma:n) = P \cup (0:\Gamma:n)$$

by Corollary 3.9 hence $m \in P$ since $m\gamma n \neq 0$, as required.

Lemma 3.11. Let $N = N_1 \times N_2$, where each N_i is a Γ-near-ring with identity. Then the following hold:

(i) If A is an ideal of N_1, then $\sqrt{A \times N_2} = \sqrt{A} \times N_2$.

(ii) If A is an ideal of N_2, then $\sqrt{N_1 \times A} = N_1 \times \sqrt{A}$.

Proof. The proof is straightforward.

Theorem 3.12. Let $N = N_1 \times N_2$, where each N_i is a Γ-near-ring with identity. If P is a weakly completely primary (completely primary) ideal of N_1, then $P \times N_2$ is a weakly completely primary (completely primary) ideal of N.

Proof. Suppose that $N = N_1 \times N_2$, where each N_i is a Γ-near-ring with identity and P is a weakly completely primary ideal of N_1. Let

$$0 \neq (a,b)\gamma(c,d) = (a\gamma c, b\gamma d) \in P \times \mathrm{N}_2,$$

where $(a,b),(c,d) \in N, \gamma \in \Gamma$ so either $a \in \sqrt{P}$ *or* $c \in P$ since P is weakly completely primary. It follows that either

$$(a,b) \in \sqrt{P} \times N_2 = \sqrt{P \times N_2} \text{ or } (c,d) \in P \times \mathrm{N}_2.$$

By Definition of weakly completely primary ideal, we have $P \times N_2$ is a weakly completely primary ideal of N.

Corollary 3.13. Let $N = N_1 \times N_2$, where each N_i is a Γ-near-ring with identity. If P is a weakly completely primary (completely primary) ideal of N_2, then $N_1 \times P$ is a weakly completely primary (completely primary) ideal of N.

Proof. This follows from Lemma 3.12.

Corollary 3.14. Let $N = \prod_{i=1}^{n} N_i$, where each N_i is a Γ-near-ring with identity. If P is a weakly completely primary (completely primary) ideal of N_j, then $N_1 \times N_2 \times \ldots \times P_j \times N_{j+1} \times \ldots \times N_n$ is a weakly completely primary (completely primary) ideal of N.
Proof. This follows from Theorem 3.12 and Corollary 3.13.

Theorem 3.15. Let $N = N_1 \times N_2$, where each N_i is a Γ-near-ring with identity. If P is a weakly completely primary ideal of N, then either $P = 0$ or P is completely primary.
Proof. Let $N = N_1 \times N_2$, where each N_i is a Γ-near-ring with identity and let $P = P_1 \times N_2$ be a weakly completely primary ideal of N. We can assume that $P \neq 0$. So there is an element (a,b) of P with $(a,b) \neq (0,0)$. Then $(0,0) \neq (a,1)\gamma(1,b) \in P$,
wehere $\gamma \in \Gamma$, gives either

$$(a,1) \in \sqrt{P} \text{ or } (1,b) \in \sqrt{P} = \sqrt{P_1} \times N_2.$$

If $(a,1) \in P$, then $P = P_1 \times \mathrm{N}_2$. We show that P_1 is completely primary hence P is weakly completely primary by Theorem 3.12. Let $c\gamma d \in P_1$, where $c,d \in N_1$. Then

$$(0,0) \neq (c,1)\gamma(d,1) = (c\gamma d,1) \in P,$$

so either $(c,1) \in P$ or $(d,1) \in \sqrt{P} = \sqrt{P_1} \times N_2$ and hence either $c \in P_1$ or $d \in P_1$. By a similar argument, $\mathrm{N}_1 \times P_2$ is completely primary.

Proposition 3.16. Let $A \subseteq P$ be proper ideals of a Γ-near-ring N. Then the following hold:

(i) If P is weakly completely primary (completely primary), then P/A is weakly completely primary (completely primary).

(ii) If A and P/A are weakly completely primary (completely primary), then P is weakly completely primary (completely primary).

Proof. (i) Let $0 \neq (a+A)\gamma(b+A) = a\gamma b + A \in P/A$, where $a,b \in N, \gamma \in \Gamma$ so $ab \in P$. If $a\gamma b = 0 \in A$, then $(a+A)\gamma(b+A) = 0$, a contradiction. So if P is weakly completely primary, then either $a \in P$ or $b \in \sqrt{P}$, hence either $a + A \in P/A$or $b + A \in \sqrt{P/A}$, as required.
(ii) Let $0 \neq a\gamma b \in P$, where $a,b \in N$, so $(a+A)\gamma(b+A) \in P/A$. For $a\gamma b \in A$, if A is weakly completely primary, then either $a \in A \subseteq P$ or $b \in A \subseteq P \subseteq \sqrt{P}$. So we may assume that $a\gamma b \notin A$. Then either $a + A \in P/A$ or $b + A \in \sqrt{P/A}$. It follows that either $a \in P$ or $b \in \sqrt{P}$ as needed.

Theorem 3.17. Let P and Q be weakly completely primary ideals of a Γ-near-ring N that are not completely primary. Then $P+Q$ is a weakly completely primary ideal of N.
Proof. Since $(P+Q)/Q \approx Q/(P \cap Q)$, we get that $(P+Q)/Q$ is weakly completely primary by Proposition 3.16 (i). Now the assertion follows from Proposition 3.16 (ii).

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