On Application of Lyapunov and Yoshizawa's Theorems on Stability, Asymptotic Stability, Boundaries and Periodicity of Solutions of Duffing's Equation

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ABSTRACT--- ***Stability is one of the properties of solutions of any differential systems. A dynamical system in a state of equilibrium is said to be stable. In other words, a system has to be in a stable state before it can be asymptotically stable which means that stability does not necessarily imply asymptotic stability but asymptotic stability implies stability. For a system to be stable depends on the form and the space for which the system is formulated. Results are available for boundedness and periodicity of solutions of second order non-linear ordinary differential equation. However, the issue of stability, asymptotic stability, with boundedness and periodicity of solutions of Duffing's equation is rare in literature. In this paper, our objective is to investigate the stability, asymptotic stability, boundedness and periodicity of solutions of Duffings equation. We employed the Lyapunov theorems with some peculiarities and some exploits on the first order equivalent systems of a scalar differential equation to achieve asymptotic stability and hence stability of Duffings equation and again using Yoshizawas theorem we proved boundedness and periodicity of solutions of a Duffings equation. Furthermore, we use fixed point technique and integrated equation as the mode to confirm apriori-bounds in achieving periodicity and boundedness of the solution.***

The results obtained showed the consequences of the cyclic relationship between different properties of solutions because the asymptotic stability converges uniformly to a point and limit of the supremum of the absolute value of the difference between the distances existed and are unique and it is this uniqueness that implies the existence of stability. The space where this existed is the space which confirmed continuous closed and bounded nature of the solution and hence the existence of optimal solution and opened the window for application of abstract implicit function theorem in Banach'sSpace to guarantee uniqueness and asymptotic stability, ultimate boundedness and periodicity of solutions of Duffings equation. We concluded that the objectives for the paper were achieved based on our deductions.

Keywords---Lyapunov theorems, Yoshizawa's theorem, stability, asymptotic stability, boundedness, periodicity Duffing's equation.

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1. INTRODUCTION

[19] Opined that the differential equation which describes a non-linear oscillator first introduced by Duffing with cubic stiffness constant has become a very common example of a non-linear oscillator. This equation permits the description of hand spring and remains of continuous interest for example: In a family of planar maps, depending on parameters, the onset of chaos typically occurs at the parameter values for the stable and unstable manifolds of a stable point come into contact tangentially. This method of creation of transversal homoclinic points and related issues can be established by the general form of which is

$$\ddot{x}(t) + \delta\dot{x}(t) - x(t) + \beta x^3(t) = f(t) \qquad 1.1$$

Where f(t) is one of the following two functions $f(t) = \gamma coswt; f(t) = \gamma sinwt$. This provides a model.

"The problem of existence as well as multiplicity of periodic solutions of the forced Duffing's equation

$$\ddot{x} + g(x) + cx = f(t) \qquad 1.2$$

has been object of many works in both undamped $(C = 0)$ and damped case. Results are available for boundedness and periodicity of solutions for second order non-linear ordinary differential equations. However, the issue of stability and asymptotic of solutions with boundedness and periodicity of Duffing's equation is rare in literature.

The general approach to the stability of periodic solutions is related to the classical Lyaponov theorems based on linear approximations. This reduces the stability study of periodic solutions to the stability of system linearized at the periodic motion. Since linearized systems contain periodic coefficients; the theory of parametric resonance can be applied. Such approach with the analysis of Floquet multipliers is used in Njoku and Omari [17].

The other traditional approach to the study of stability of periodic solutions is related to approximate average and multiple scales method which reduces original time dependent dynamical systems to autonomous system. In this case, stability study is reduced to analysis of fixed. The Existence and asymptotic stability of periodic solutions of a Duffing's equation $\ddot{x} + c\dot{x} + g(t,x) = 0$ taking advantage of a new maximum principle with L^p-conditions combined with known relations between upper and lower solutions of topological and stability were considered by Pedro, F. Tores [16]. The existence and uniqueness of solution of Duffings equation using Abstract implicit function theorem in Banach Spaces were considered by Eze, E. O ,Ugbene, I .J and Ogbu, H. M [4].

Agarwal [1] held the views that by using inequalities on the Green function and non-linearity alternatives, they obtained existence result of a conjugate or/and a focal boundary value problem (BVP) under smallness and sign assumption on f mainly if $|f(t,x,x^I,\dots,x^{n-1}| \leq \alpha(t)\varphi|x|$, $x(t)$ is stable if and only if the zero solution of $\dot{y}(t) = F(t,y)$ with $F(t,0) = 0$. Using techniques from the stability theory of differential equation [18], gave conditions on x(t) for E(t) to be upper bounded linearly or by a constant for t≥o. More concretely, these techniques give constant or have bounds on E (t) when x(t) is a trajectory of a dynamical system which falls into a stable, hyperbolic fixed point or into a stable, hyperbolic cycle or into a normally hyperbolic and contracting manifold with quasi- periodic flow on the manifold.

In this paper, our objective is to investigate the stability, asymptotic stability, boundedness and periodicity of solutions of Duffing's equation. This task will be achieved through the following:

i. The use of Lyapunov functions $V: \mathbb{R}^2 \to \mathbb{R}$ with some peculiar properties to achieve stability, asymptotic stability of Duffing's equation
ii. Yoshizawa's theorem was used to achieve boundedness and hence periodicity.
iii. The use of fixed point technique and an integrated equation as the mode for estimating the apriori bound in achieving periodicity and boundedness of solutions of Duffing's equation.

Now consider the Duffings Equation of the form:

$$\ddot{x} + ax + bx^2 + 2x^3 = p(t) \qquad 1.3$$

Where a, b, c are real constants and $p: [0,2\pi] \to E^I$ is continuous. The existence of 2π-priodic solutions of (1.3) has been investigated, that is, the solutions defined on $[0,2\pi]$ such that

$$x(0) = x(2\pi) \qquad 1.4$$

and

$$\dot{x}(0) = \dot{x}(2\pi) \qquad 1.5$$

Now equation (1.3) - (1.5) can be reduced to a more general form

$$\ddot{x} + a\dot{x} + +h(x) = p(t) \qquad 1.6$$

Subject to the boundary conditions:

$$D^{(r)}x(0) = D^{(r)}x(2\pi); \;\; r = 0,1,2, \qquad 1.7$$

where $a > 0$, and h(x), p(t) are continuous functions depending on their argument.

For constant coefficient equation

$$\ddot{x} + a\dot{x} + bx = p(t); \;\; for\; a > 0, b > 0 \qquad 1.8$$

Ezeilo (1986) has shown that if the Ruth-Hurwitz's conditions a>o,b>o hold, the roots of the ordinary equation

$$\lambda^2 + a\lambda + b = 0 \qquad 1.9$$

have negative real parts, then asymptotically stable and ultimate boundedness of solution can be verified for (1.8) when $p(t) = 0$. The existence of periodic solutions can be verified for (1.6) when (1.8) holds. Ezeilo [5] Tejumola [14], [2], Ogbu [12] Ezeilo, Ogbu [8], and Eze, et al [4].

A close look at (1.6) and (1.8) gives some clue to the theorems stated below:

2. PRELIMINARIES

Theorem **2.1**: Suppose there exists a>0, b>0, and β>0 such that:.

$$(i)\; h^1(x) < b, \beta^2 = b$$
$$(ii)\; |h(x) - x| > 0\; for\; all\; x$$
$$(iii)\; |h(x)| \to \infty\; as\; |x| \to \infty$$
$$(iv) x^2 + y^2 \to \infty\; as\; |x| \to \infty, |y| \to \infty$$

Then equation (1.6) through (1.8) has stable, bounded and periodic solutions when p(t)=0.

Theorem 2.2: Suppose further in theorem (2.1) the conditions (i) replaced by

$$(i)\; h^1(x) < b, \qquad \beta^2 \neq b, \qquad |a\dot{x} - p(t)| > 0$$

Then equation (1.6) and (1.8) has stable, bounded and periodic solutions when $p(t) \neq 0$. Consider the scalar equation

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \qquad f(0) = 0 \qquad 2.1$$

where f is sufficiently smooth.

Theorem 2.3: (Lyapunov) Assume that

$(i)\ f \in C^1$
$(ii)\ there\ exists\ a\ C^1\ function, V: \mathbb{R}^n \to \mathbb{R}$ $\quad Such\ that\ V(x) > 0\ for\ every\ x\ and\ V(x) = 0\ if\ x = 0$
$(iii)\ Along\ the\ solution\ paths\ of\ equation\ (2.1)\ \dot{V} \le 0.$
Then the solution $x = 0$ of equation (2.1) is stable in the sense of Lyaponov.

Theorem 2.4: (Lyapunov)
$(i)\ f \in C^1$
$(ii)\ there\ exists\ a\ C^1\ function, V: \mathbb{R}^n \to \mathbb{R}$
$Such\ that\ V(x) > 0\ for\ every\ x\ and\ V(x) = 0\ if\ x = 0$
$(iii)\ Along\ the\ solution\ paths\ of\ equation\ (2.1)\ \dot{V} < 0, if\ x \neq 0\ and\ \dot{V} = 0\ i.e\ \dot{V}\ is\ negative\ definite.$
Then the solution $x = 0$ of equation (2.1) is asymptotically stable the sense of Lyapunov.

Theorem 2.5: (Yoshizawa)
Let us consider the system.
$$\left.\begin{aligned} \dot{x} &= f(t, x, y), \\ \dot{y} &= g(t, x, y) \end{aligned}\right\} \qquad 2.2$$
where f, g satisfy conditions for existence of solutions for any given initial values.
Suppose there exists a function $V: \mathbb{R}^2 \to \mathbb{R}$ with first partial derivatives in its argument such that $V(x, y) \to +\infty\ as\ x^2 + y^2 \to \infty$ and such that for any solution $x(t), y(t)$ of equation (2.2)
$\dot{V} = \frac{d}{dt} V(x(t), y(t) \le -\delta < 0\ \ If\ x^2(t) + y^2(t) \ge \mathbb{R} > 0$, where δ and ℝ are finite constants.
Then every solution $x(t), y(t)$ of equation (2.2) is uniformly ultimately bounded with bounding constants depending on ℝ and now
$V \to +\infty\ as\ x^2(t) + y^2(t) \to \infty$
The conclusion here is that there exists a constant D, $(0 < D < \infty)\ such\ that\ |x(t)| \le D, |y(t)| \le D$

Definition 2.6: A continuous function $V(x, t) = V(x_1, x_2, \dots, x_n, t)$ is called positive definite if $\lim_{\|x\| \to 0} V(x, t) = 0$ and there exist $\varphi\|x\|$ such that$V(x, t) \ge \varphi\|x\|$. The function $\varphi\|x\|$ must be monotonically increasing function in $\|x\|$and $\varphi(0) = 0$.

Definition 2.7: The function $V(x, t) = V(x_1, x_2, \dots, x_n, t)$ is called negative definite if there exists $\varphi\|x\|$ of the type described such that$V(x, t) \le -\varphi\|x\|$.

3. MAIN RESULT

Here the proof of theorem (2.1) entails establishing stability, boundedness and periodicity for our equations (1.6 - 1.7) when p(t)=0 that is
$$\ddot{x} + a\dot{x} + h(x) = 0 \qquad 3.1$$
or the equivalent system
$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -ay - h(x) \end{aligned}\right\} \qquad 3.2$$
Consider the function $V: \mathbb{R}^2 \to \mathbb{R}$ defined by
$$V = \frac{1}{2}y^2 + H(x) \qquad 3.3$$
where $H(x) = \int_0^x h(s)ds$
Clearly the V as defined above is positive semi-definite. The time derivative $\dot{V}$ along the solution paths of (4.2) is

$$\begin{aligned} \dot{V} &= y\dot{y} + h(x)\dot{x} \\ &= y\big(-ay - h(x)\big) + h(x)y \\ &= -ay^2 - h(x)y + h(x)y \\ &= -ay^2 \end{aligned}$$
This is negative definite. There by Lyapunov theorem the system (3.1) - (3.2) is asymptotically stable. Hence it is stable. Therefore the system (3.1) - (3.2) is stable in the sense of Lyapunov when p(t)=0.
Now for the proof of boundedness in equation (3.1) and (3.2); let us consider the C^1 function, here $V: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$V = \frac{1}{2}x^2 + \frac{1}{2}y^2 \qquad 3.4$$

The V defined in equation (4.4) is positive semi-definite. The time derivative V along the solution paths of (3.2) is

$$\begin{aligned}\dot{V} &= x\dot{x} + y\dot{y}\\ &= xy - ay^2 - yh(x)\\ &= ay^2 - y(h(x) - x)\end{aligned}$$

Since $|h(x) - x| > 0$ for all x (condition (ii) in theorem (2.1) then

$$\dot{V} = -ay^2 - y|h(x) - x| < 0 \qquad 3.5$$

without loss of generality, V is such that $\dot{V} = -1\ since\ x^2 + y^2 \to \infty\ as\ |x| \to \infty, |y| \to \infty$

By Yoshizawa's theorem equation (3.1) has a bounded solution. Therefore equation (1.6) has bounded solutions when p(t)=0.

Now the condition (i) in theorem (2.1) which is $\beta^2 = b$ implies that $i\beta$ is a root of the auxiliary equation. Therefore the solution to (4.1) is of the form $Acos\beta(t) + Bsin\beta(t)$. This clearly shows that the solution is periodic. Therefore equation (4.1) is stable, bounded and periodic. Here the proof of theorem (2.1).

The proof of theorem (2.2) is as follows:

Consider equation (1.6) or its equivalent system

$$\left.\begin{aligned}\dot{x} &= y\\ \dot{y} &= -ay - h(x) + p(t)\end{aligned}\right\} \qquad 3.6$$

and the function $V: \mathbb{R}^2 \to \mathbb{R}$ defined by

$$V = \tfrac{1}{2}y^2 + H(x) \qquad 3.7$$

where $H(x) = \int_0^x h(s)ds$

Clearly the V defined above in equation (3.7) is positive semi-definite. The time derivative $\dot{V}$ along the solution paths of (4.6) is

$$\begin{aligned}\dot{V} &= y\dot{y} + h(x)\dot{x}\\ &= y(-ay - h(x) + p(t) + h(x)y)\\ &= -ay^2 - yh(x) + yp(t) + h(x)y\\ &= -ay^2 + yp(t)\\ &= -y(ay - p(t)\\ &= -y\big(a\dot{x} - p(t)\big) < 0\ for\ |a\dot{x} - p(t)| > 0\end{aligned}$$

This is a negative definite.

Therefore by Lyapunov theorem, the system (3.6) is asymptotically stable in the sense of Lyapunov and hence stable.

Next we proceed to establish boundedness and periodicity in equation (1.6) a parameter λ, dependent equation.

$$\ddot{x} + a\dot{x} + h\lambda(x) = \lambda p(t) \qquad 3.8$$

where

$$h\lambda(x) = (1 - \lambda)bx + \lambda h(x) \qquad 3.9$$

where λ is in the range $0 \le \lambda \le 1$ and b is a constant satisfying (1.6). The equation (4.8) is equivalent to

$$\left.\begin{aligned}\dot{x} &= y\\ \dot{y} &= -ay - h\lambda(x) - pt\end{aligned}\right\} \qquad 3.10$$

The system equation (4.10) can be represented in the vector for

$$\dot{x} = AX + \lambda F(t, x)$$

Where

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}\ F = |p(t) - h(t) + bx|$$

We remark that equation (4.8) reduces to a linear equation

$$\ddot{x} + a\dot{x} + bx = 0 \qquad 3.11$$

Where λ=0 and to equation (1.6) when λ=1

If the roots of the auxiliary equation (4.11) has no roots of the form

$$\beta^2 \neq b, \beta^2 \neq 0 \qquad 3.12$$

(β an integer), then equation (1.6) and (1.7) has at least one 2π periodic solution that is the matrix $(e^{-2\pi}, -1)$ where 1 is the identity 2x2 matrix is invertible. Therefore x is a 2π periodic solution of equation (4.11) if and only if

$$x = \lambda, TX,\ \ 0 \le \lambda \le 1 \qquad 3.13$$

$$TX(t) = \int_0^{2\pi} (e^{-2\pi A} - 1)e^{(t-s)A}F\big(s, X(s)\big)ds \qquad 3.14$$

Jack Hale 1963
Let S be the space of all real valued continuous 2π vector function $X(t) = (\bar{X}(t), \bar{Y}(t))$ which are of period2π. If the mapping T is completely continuous mapping of S into itself. Then existence of a 2π-periodic solution (1.6) - (1.7) correspond to $X \in S$ satisfying equation (4.13) for λ=1. Finally using Lemma [13] established that
$|x|_\infty \leq C_6 \; and \; |\dot{x}|_\infty \leq C_3$ where the C's are the apriori bounds.

4. DISCUSSION

I. The solutions of a differential equation need not converge to a point $\bar{x}$ for $\bar{x}$ to be stable but must remain sufficiently close to $\bar{x}$ for all$t \geq t_0$. A steady state is asymptotically stable if it is stable and converges to the point $\bar{x}$ as$t \to \infty$. [15]

II. We noted that stability does not necessarily imply asymptotic stability but that asymptotic stability implies stability. This is precisely because the limit of the supremum of the distances between the two points must exist and it is unique and uniqueness of the solution implies existence of the solution but the converse is not true. Again Boundedness being one of the properties of uniform convergence occurs because the asymptotic stability converges uniformly to a point and that point is the optimal point and it is unique, bounded and closed and a fixed point which coincides with the optimal solution. Again the sphere where the stability existed must exist for the asymptotic stability but must return to the origin. The distance of the diameter of the sphere must be finite and thus bounded. We stress that a solution must exist before we talk of asymptotic stability and hence stability. This optimal solution must occur in a closed and bounded interval. [Refer to Picard theorem]

III. A Lyapunov function relative to a set G defined on a set E (closed) which under the conditions of the theorem contains (locates) all the positive limit sets of solutions which for positive remain in G

IV. . If X(t) is a solution of our differential equation (1.2), in fact if X(t) is any continuous function in R to, nR then its positive limit set is closed and connected. If X(t) is bounded, then its positive limit set is compact.

V. If V is a Lyapunov function on G for the periodic system, then each solution of the system that is bounded and remains in G for all t>0(t<0) approaches M as t→∞ (t→-∞)

VI. There are also some special classes of non-autonomous systems where the limit sets of the solution have an invariance property. The simplest of these are periodic systems.

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