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The Solutions and Periods of Two Non-linear Discrete Models

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ABSTRACT— ***We analyse the solutions of the a non-linear discrete models in this study. Then we obtain the period of solutions of these systems.***

Keywords— Discrete model, discrete systems, period of discrete model, solution of discrete model

1. INTRODUCTION

Non-linear difference models have been taken into consideration in many branches of mathematics as well as other sciences for years. Because of derivation of many complex behavior based on simple formulation, it is a very interesting subject. It has many applications in many science branches and it is easy to understand. This tutorial is to introduce a simple difference equation system, especially the four important subjects: the equilibrium points of this system, the period of this system, its solution and its dynamics [1], [2], [3], [4]. One can encounter many investigations and interest in the field of functions of difference equations. Some of them are follows. In [5], [6], Cınar C., Yalcinkaya I. and Iricanin B., Stevic S. took into consideration some systems of non-linear difference equations of higher order with periodic solutions. Nasri M and at all, introduced a deterministic model for HIV infection in the presence of combination therapy related to difference equations system [2]. Clark and Kulenovic, in [1], investigated the global stability properties and asymptotic behavior of solutions of the recursive equations system. We obtained the equilibrium points of some non-linear discrete systems and examinated stability and dynamics of these systems in [3]. Kılıklı, [4], studied on some non-linear discrete systems, periodicity and stability of these systems illustrated in your study.

In this study, we consider the following difference equation systems

$$x_{n+1}=\frac{A}{y_{n-1}},\ y_{n+1}=\frac{B}{z_{n-1}-x_{n-1}},\ z_{n+1}=\frac{Ax_{n-1}}{x_n y_{n-2}}+\frac{A}{y_{n-1}},(n\geq 0) \tag{1.1}$$

with initial values $x_{-1},x_0\,,y_{-2},y_{-1},y_0,z_{-1},z_0\in\ \mathbb{R}-\{0\}, z_{-1}-x_{-1}\neq 0, z_0-x_0\neq 0,\ \mathrm{A},\mathrm{B}\in\mathbb{R}-\{0\}$ and

$$x_{n+1}=\frac{A}{y_{n-2}},\ y_{n+1}=\frac{B}{z_{n-2}-x_{n-2}},\ z_{n+1}=\frac{Ax_{n-1}}{x_n y_{n-3}}+\frac{A}{y_{n-2}},(n\geq 0) \tag{1.2}$$

with initial values $x_{-2},x_{-1},x_0\,,y_{-3},y_{-2},y_{-1},y_0,z_{-2},z_{-1},z_0\in\ \mathbb{R}-\{0\},\ z_{-2}-x_{-2}\neq 0,z_{-1}-x_{-1}\neq 0,z_0-x_0\neq 0$, $\mathrm{A},\mathrm{B}\in\mathbb{R}-\{0\}$.

Firstly, we give basic preliminary definitions and a theorem. Let I_1,I_2 and I_3 be some intervals of real numbers and let $F_1\colon I_2\to I_1,F_2\colon I_3\times I_1\to I_2,F_3\colon I_2\times I_1\to I_3$ be three continuously differentiable functions. For every initial condition $(x_s,y_s,z_s)\in I_1\times I_2\times I_3$, it is obvious that the system of difference equations (1.3)

$$x_{n+1}=F_1(y_n),\ y_{n+1}=F_2(x_n,z_n),\ z_{n+1}=F_3(x_n,y_n) \tag{1.3}$$

has a unique solution $\{x_n,y_n,z_n\}$.

A solution $\{x_n,y_n,z_n\}$ of the system of difference equations (1.3) is periodic if there exist a positive integer p such that

$$x_{n+p}=x_n,\qquad y_{n+p}=y_n,\qquad z_{n+p}=z_n$$

the smallest such positive integer p is called the prime period of the solution of difference equation system (1.3). A point $(x,y,z)\in I_1\times I_2\times I_3$ is called an equilibrium point of system (1.3), if

$$x=F_1(y),\qquad y=F_2(x,z),\qquad z=F_3(x,y).$$

2. THE SOLUTIONS AND PERIODS OF SOME DISCRETE MODELS

In this section all results have been obtained by using [3], [4], **[5]SONMAKALE**. The following theorems show us the period of solutions of the systems (1.1) and (1.2).

Theorem 2.1. Suppose that $\{x_n, y_n, z_n\}$ are the solutions of the difference equation system (1.1) with initial values $x_{-1}, x_0, y_{-2}, y_{-1}, y_0, z_{-1}, z_0 \in \mathbb{R}-\{0\}, z_{-1}-x_{-1} \neq 0, z_0 - x_0 \neq 0$. Then all solutions of the system (1.1) are periodic with period 6.

Proof: From the system (1.1), it is obtained the following equalities

$$x_{n+1} = \frac{A}{y_{n-1}}, \qquad y_{n+1} = \frac{B}{z_{n-1}-x_{n-1}}, \qquad z_{n+1} = \frac{Bx_{n-1}}{x_n y_{n-2}} + \frac{A}{y_{n-1}},$$

$$x_{n+2} = \frac{A}{y_n}, \qquad y_{n+2} = \frac{B}{z_n - x_n}, \qquad z_{n+2} = \frac{Bx_n}{A} + \frac{A}{y_n},$$

$$x_{n+3} = \frac{A(z_{n-1}-x_{n-1})}{B}, \qquad y_{n+3} = \frac{x_n y_{n-2}}{x_{n-1}}, \qquad z_{n+3} = \frac{A(z_{n-1}-x_{n-1})}{B} + \frac{B}{y_{n-1}},$$

$$x_{n+4} = \frac{A(z_n - x_n)}{B}, \qquad y_{n+4} = \frac{A}{x_n}, \qquad z_{n+4} = \frac{A(z_n - x_n)}{B} + \frac{B}{y_n},$$

$$x_{n+5} = \frac{Ax_{n-1}}{x_n y_{n-2}}, \qquad y_{n+5} = y_{n-1}, \qquad z_{n+5} = \frac{Ax_{n-1}}{x_n y_{n-2}} + (z_{n-1}-x_{n-1}),$$

$$x_{n+6} = x_n, \qquad y_{n+6} = y_n, \qquad z_{n+6} = z_n.$$

Thus all solutions of the system (1.1) are periodic with 6 period.

Theorem 2.2. All solutions of the difference equation system (1.1) with initial values $x_{-1} = s, x_0 = t, y_{-2} = p, y_{-1} = q, y_0 = r, z_{-1} = k, z_0 = l \in \mathbb{R}-\{0\}, k-s \neq 0, l-t \neq 0$, follow

$$x_{6n+1} = \frac{A}{q}, \qquad y_{6n+1} = \frac{B}{k-s}, \qquad z_{6n+1} = \frac{Bs}{tp} + \frac{A}{q},$$

$$x_{6n+2} = \frac{A}{r}, \qquad y_{6n+2} = \frac{B}{l-t}, \qquad z_{6n+2} = \frac{Bt}{A} + \frac{A}{r},$$

$$x_{6n+3} = \frac{A(k-s)}{B}, \qquad y_{6n+3} = \frac{tp}{s}, \qquad z_{6n+3} = \frac{A(k-s)}{B} + \frac{B}{q},$$

$$x_{6n+4} = \frac{A(l-t)}{B}, \qquad y_{6n+4} = \frac{A}{t}, \qquad z_{6n+4} = \frac{A(l-t)}{B} + \frac{B}{r},$$

$$x_{6n+5} = \frac{As}{tp}, \qquad y_{6n+5} = q, \qquad z_{6n+5} = \frac{As}{tp} + (k-s),$$

$$x_{6n+6} = t, \qquad y_{6n+6} = r, \qquad z_{6n+6} = l.$$

Proof: By using induction method, it is obvious that above results hold for $n = 0$. Assume that these equalities hold. Now we must show that above results hold for $n = s + 1$.

$$x_{6s+7} = \frac{A}{y_{6s+5}} = \frac{A}{q}, \qquad y_{6s+7} = \frac{B}{z_{6s+5} - x_{6s+5}} = \frac{B}{k-s}, \qquad z_{6s+7} = \frac{Bx_{6s+5}}{x_{6s+6}y_{6s+4}} + \frac{A}{y_{6s+5}} = \frac{Bs}{tp} + \frac{A}{q},$$

$$x_{6s+8} = \frac{A}{y_{6s+6}} = \frac{A}{r}, \qquad y_{6s+8} = \frac{B}{z_{6s+6} - x_{6s+6}} = \frac{B}{l-t}, \qquad z_{6s+8} = \frac{Bx_{6s+6}}{x_{6s+7}y_{6s+5}} + \frac{A}{y_{6s+6}} = \frac{Bt}{A} + \frac{A}{r},$$

$$x_{6s+9} = \frac{A}{y_{6s+7}} = \frac{A}{B}(k-s), \qquad y_{6s+9} = \frac{B}{z_{6s+7} - x_{6s+7}} = \frac{tp}{s}, \qquad z_{6s+9} = \frac{Bx_{6s+7}}{x_{6s+8}y_{6s+6}} + \frac{A}{y_{6s+7}} = \frac{A}{B}(k-s) + \frac{B}{q},$$

$$x_{6s+10} = \frac{A}{y_{6s+8}} = \frac{A}{B}(l-t), \qquad y_{6s+10} = \frac{B}{z_{6s+8} - x_{6s+8}} = \frac{A}{t}, \qquad z_{6s+10} = \frac{Bx_{6s+8}}{x_{6s+9}y_{6s+7}} + \frac{A}{y_{6s+8}} = \frac{A}{B}(l-t) + \frac{B}{r},$$

$$x_{6s+11} = \frac{A}{y_{6s+9}} = \frac{As}{tp}, \qquad y_{6s+11} = \frac{B}{z_{6s+9} - x_{6s+9}} = q, \qquad z_{6s+11} = \frac{Bx_{6s+9}}{x_{6s+10}y_{6s+8}} + \frac{A}{y_{6s+9}} = \frac{As}{tp} + (k-s),$$

$$x_{6s+12} = \frac{A}{y_{6s+10}} = t, \qquad y_{6s+12} = \frac{B}{z_{6s+10} - x_{6s+10}} = r, \qquad z_{6s+12} = \frac{Bx_{6s+10}}{x_{6s+11}y_{6s+9}} + \frac{A}{y_{6s+10}} = l.$$

Theorem 2.3. Suppose that $\{x_n, y_n, z_n\}$ are the solutions of the difference equation system (1.2) with initial values $x_{-2}, x_{-1}, x_0, y_{-3}, y_{-2}, y_{-1}, y_0, z_{-2}, z_{-1}, z_0 \in \mathbb{R} - \{0\}, z_{-2} - x_{-2} \neq 0, z_{-1} - x_{-1} \neq 0, z_0 - x_0 \neq 0$. Then all solutions of the system (1.2) are periodic with period 8.

Proof: From the system (1.2), it is obtained the following equalities

$$x_{n+1} = \frac{A}{y_{n-2}}, \qquad y_{n+1} = \frac{B}{z_{n-2} - x_{n-2}}, \qquad z_{n+1} = \frac{Bx_{n-1}}{x_n y_{n-3}} + \frac{A}{y_{n-2}},$$

$$x_{n+2} = \frac{A}{y_{n-1}}, \qquad y_{n+2} = \frac{B}{z_{n-1} - x_{n-1}}, \qquad z_{n+2} = \frac{Bx_n}{A} + \frac{A}{y_{n-1}},$$

$$x_{n+3} = \frac{A}{y_n}, \qquad y_{n+3} = \frac{B}{z_n - x_n}, \qquad z_{n+2} = \frac{B}{y_{n-2}} + \frac{A}{y_n},$$

$$x_{n+4} = \frac{A(z_{n-2} - x_{n-2})}{B}, \qquad y_{n+4} = \frac{x_n y_{n-3}}{x_{n-1}}, \qquad z_{n+4} = \frac{A(z_{n-2} - x_{n-2})}{B} + \frac{B}{y_{n-1}},$$

$$x_{n+5} = \frac{A(z_{n-1} - x_{n-1})}{B}, \qquad y_{n+5} = \frac{A}{x_n}, \qquad z_{n+5} = \frac{A(z_{n-1} - x_{n-1})}{B} + \frac{B}{y_n},$$

$$x_{n+6} = \frac{A(z_n - x_n)}{B}, \qquad y_{n+6} = y_{n-2}, \qquad z_{n+6} = \frac{A(z_n - x_n)}{B} + (z_{n-2} - x_{n-2}),$$

$$x_{n+7}=\frac{Ax_{n-1}}{x_n y_{n-3}},\qquad y_{n+7}=y_{n-1},\qquad z_{n+7}=\frac{Ax_{n-1}}{x_n y_{n-3}}+(z_{n-1}-x_{n-1}),$$

$$x_{n+8}=x_n,\qquad y_{n+8}=y_n,\qquad z_{n+8}=z_n.$$

Thus all solutions of the system (1.2) are periodic with 8 period.

Theorem 2.4. All solutions of the difference equation system (1.2) with initial values $x_{-2}=k, x_{-1}=l, x_0=m\,, y_{-3}=p, y_{-2}=q, y_{-1}=r, y_0=u, z_{-2}=t, z_{-1}=h, z_0=g\in\ \mathbb{R}-\{0\}, t-k\neq 0, h-l\neq 0, g-m\neq 0$ follow

$$x_{8n+1}=\frac{A}{q},\qquad y_{8n+1}=\frac{B}{t-k},\qquad z_{8n+1}=\frac{Bl}{mp}+\frac{A}{q},$$

$$x_{8n+2}=\frac{A}{r},\qquad y_{8n+2}=\frac{B}{h-l},\qquad z_{8n+2}=\frac{Bm}{A}+\frac{A}{r},$$

$$x_{8n+3}=\frac{A}{u},\qquad y_{8n+3}=\frac{B}{g-m},\qquad z_{8n+3}=\frac{A}{u}+\frac{B}{q},$$

$$x_{8n+4}=\frac{A(t-k)}{B},\qquad y_{8n+4}=\frac{mp}{l},\qquad z_{8n+4}=\frac{A(t-k)}{B}+\frac{B}{r},$$

$$x_{8n+5}=\frac{A(h-l)}{B},\qquad y_{8n+5}=\frac{A}{m},\qquad z_{8n+5}=\frac{A(h-l)}{B}+\frac{B}{u},$$

$$x_{8n+6}=\frac{A(g-m)}{B},\qquad y_{8n+6}=q,\qquad z_{8n+6}=\frac{A(g-m)}{B}+(t-k),$$

$$x_{8n+7}=\frac{Al}{mp},\qquad y_{8n+7}=r,\qquad z_{8n+7}=\frac{Al}{mp}+(h-l),$$

$$x_{8n+8}=m,\qquad y_{8n+8}=u,\qquad z_{8n+8}=g,$$

Proof: By using induction method, it is obvious that above results hold for $n=0$. Assume that these equalities hold for $n=s$. Now we must show that above results hold for $n=s+1$.

$$x_{8s+9}=\frac{A}{y_{8s+6}}=\frac{A}{k},\qquad y_{8s+9}=\frac{B}{z_{8s+6}-x_{8s+6}}=\frac{B}{t-k},\qquad z_{8s+9}=\frac{Bx_{8s+7}}{x_{8s+8}y_{8s+5}}+\frac{A}{y_{8s+6}}=\frac{Bl}{mp}+\frac{A}{q},$$

$$x_{8s+10}=\frac{A}{y_{8s+7}}=\frac{A}{r},\qquad y_{8s+10}=\frac{B}{z_{8s+7}-x_{8s+7}}=\frac{B}{h-l},\qquad z_{8s+10}=\frac{Bx_{8s+8}}{x_{8s+9}y_{8s+6}}+\frac{A}{y_{8s+7}}=\frac{Bm}{A}+\frac{A}{r},$$

$$x_{8s+11}=\frac{A}{y_{8s+8}}=\frac{A}{u},\quad y_{8s+11}=\frac{B}{z_{8s+8}-x_{8s+8}}=\frac{B}{g-m},\quad z_{8s+11}=\frac{Bx_{8s+9}}{x_{8s+10}y_{8s+7}}+\frac{A}{y_{8s+8}}=\frac{A}{u}+\frac{B}{q},$$

$$x_{8s+12}=\frac{A}{y_{8s+9}}=\frac{A(t-k)}{B},\quad y_{8s+12}=\frac{B}{z_{8s+9}-x_{8s+9}}=\frac{mp}{l},\quad z_{8s+12}=\frac{Bx_{8s+10}}{x_{8s+11}y_{8s+8}}+\frac{A}{y_{8s+9}}=\frac{A(t-k)}{B}+\frac{B}{r},$$

$$x_{8s+13}=\frac{A}{y_{8s+10}}=\frac{A(h-l)}{B},\quad y_{8s+13}=\frac{B}{z_{8s+10}-x_{8s+10}}=\frac{A}{m},\quad z_{8s+13}=\frac{Bx_{8s+11}}{x_{8s+12}y_{8s+9}}+\frac{A}{y_{8s+10}}=\frac{A(h-l)}{B}+\frac{B}{u},$$

$$x_{8s+14}=\frac{A}{y_{8s+11}}=\frac{A(g-m)}{B}, y_{8s+14}=\frac{B}{z_{8s+11}-x_{8s+11}}=q, z_{8s+14}=\frac{Bx_{8s+12}}{x_{8s+13}y_{8s+10}}+\frac{A}{y_{8s+11}}=\frac{A(g-m)}{B}+t-k,$$

$$x_{8s+15}=\frac{A}{y_{8s+12}}=\frac{Al}{mp},\quad y_{8s+15}=\frac{B}{z_{8s+12}-x_{8s+12}}=r,\quad z_{8s+15}=\frac{Bx_{8s+13}}{x_{8s+14}y_{8s+11}}+\frac{A}{y_{8s+12}}=\frac{Al}{mp}+(h-l),$$

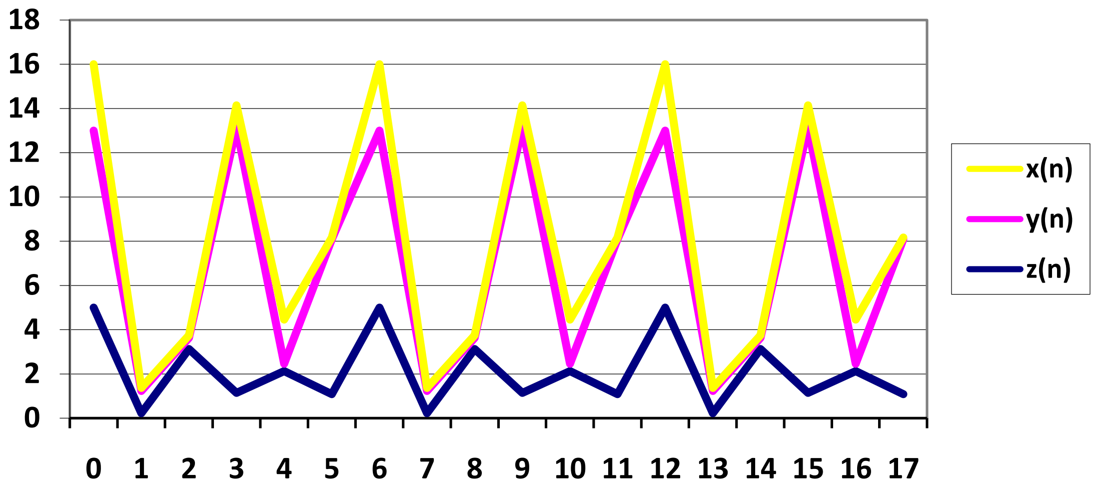
$$x_{8s+16}=\frac{A}{y_{8s+13}}=m,\quad y_{8s+16}=\frac{B}{z_{8s+13}-x_{8s+13}}=u,\quad z_{8s+16}=\frac{Bx_{8s+14}}{x_{8s+15}y_{8s+12}}+\frac{A}{y_{8s+13}}=g.$$

3. ILLUSTRATIVE EXAMPLES

Example 3.1. Let be $x_{-1}=1, x_0=3$, $y_{-2}=4, y_{-1}=7, y_0=8, z_{-1}=2, z_0=5$, $A=1, B=1$ in (1.1). In this case the solutions of (1.1) are periodic with 6.

n	x_n	y_n	z_n
0	3	8	5
1	0,142857	1	0,226190
2	0,125	0,5	3,125003
3	1	12,000048	1,142857
4	2	0,333333	2,125
5	0,083333	7,000007	1,083333
6	3	8	5
7	0,142857	1	0,226190
8	0,125	0,5	3,125003
9	1	12,000048	1,142857
10	2	0,333333	2,125
11	0,083333	7,000007	1,083333
12	3	8	5
13	0,142857	1	0,226190
14	0,125	0,5	3,125003
15	1	12,000048	1,142857
16	2	0,333333	2,125
17	0,083333	7,000007	1,083333
18	3	8	5

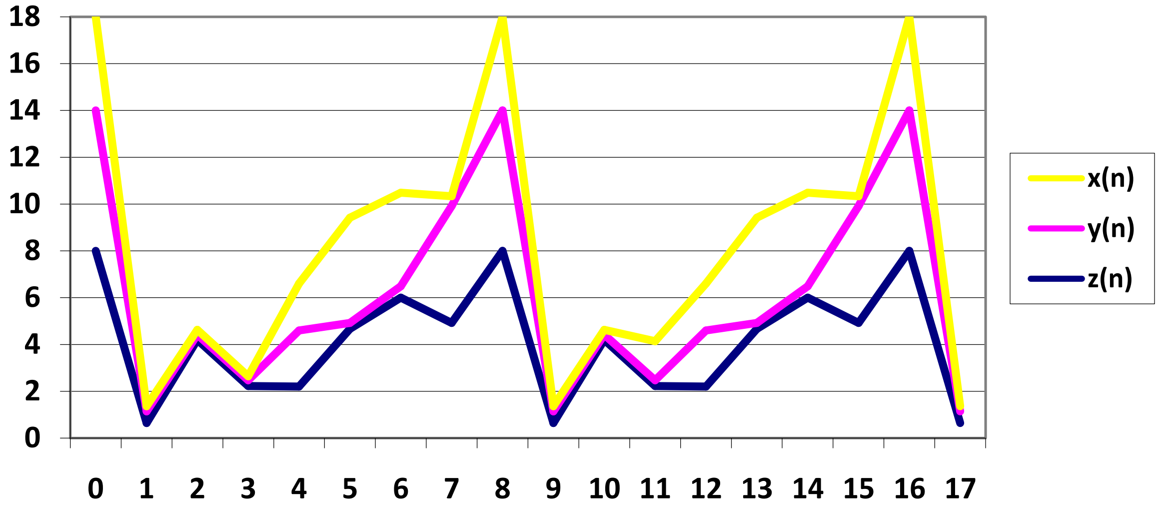
Figure 3.1.



Examples 3.2. Let be $x_{-2} = 1, x_{-1} = 2.5, x_0 = 4$, $y_{-3} = 1.5, y_{-2} = 4.5, y_{-1} = 5, y_0 = 6, z_{-2} = 3, z_{-1} = 7, z_0 = 8$, $A = 1, B = 1$ in (1.1). In this case the solutions of (1.1) are periodic with 8.

n	x_n	y_n	z_n
0	4	6	8
1	0,222222	0,5	0,638888
2	0,2	0,222222	4,2000004
3	0,166666	0,25	2,222222
4	2	2,400003	2,2
5	4,500004	0,249999	4,66667
6	4	0,486486	6
7	0,416666	5	4,91667
8	4	6	8
9	0,222222	0,5	0,638888
10	0,2	0,222222	4,200004
11	0,166666	0,25	2,222222
12	2	2,400003	2,2
13	4,500004	0,249999	4,66667
14	4	0,486486	6
15	0,416666	5	4,91667
16	4	6	8
17	0,222222	0,5	0,638888
18	0,2	0,222222	4,200004

Figure 3.2.



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