Some Basic Properties of Quasi-Primary and Primary Ideals in AG-Groupoids

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ABSTRACT— ***The purpose of this paper is to introduce the notion of a primary and quasi- primary ideals in AG-groupoids, we study primary and quasi primary ideals in AG-groupoids. Some characterizations of primary and quasi primary ideals are obtained. Moreover, we investigate the relationships between primary and quasi primary ideals in AG-groupoids.Finally, we obtain*** *the* ***necessary and sufficient conditions of a primary ideal to be a quasi primary ideal in AG-groupoids.***

Keywords—AG-groupoid, LA-semigroup, ideal, quasi-primary ideal, primary ideal.

1. INTRODUCTION

A groupoid S is called an Abel-Grassmann's groupoid, abbreviated as an AG-groupoid, if its elements satisfy the left invertive law [1, 2], that is: $(ab)c = (cb)a$ for all $a,b,c \in S$. Several examples and interesting properties of AG-groupoids can be found in [3], [4], [5] and [6]. It has been shown in [3] that if an AG-groupoid contains a left identity then it is unique. It has been proved also that an AG-groupoid with right identity is a commutative monoid, that is, a semigroup with identity element. It is also known [2] that in an AG-groupoid S, the medial law, that is,

$$(ab)(cd) = (ac)(bd)$$

for all $a,b,c,d \in S$ holds. An AG-groupoid S is called AG-3-band [7] if its every element satisfies $a(aa) = (aa)a = a.$

Now we define the concepts that we will used. Let S be an AG-groupoid. By an AG-subgroupoid of S [8], we means a non-empty subset A of S such that $A^2 \subseteq A.$ A non-empty subset A of an AG-groupoid S is called a left (right) ideal of S [7] if $SA \subseteq A$ ($AS \subseteq A$). By two-sided ideal or simply ideal, we mean a non-empty subset of an AG- groupoid S which is both a left and a right ideal of S. A proper ideal P of an AG-groupoid S is called prime [8] if $AB \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$, for all ideals A and B in S. A proper left ideal P of an AG-groupoid S is called quasi-prime [8] if $AB \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$, for all left ideals A and B in S. It is easy to see that every quasi-prime ideal is prime.

In this paper we characterize the AG-groupoid. We investigate relationships between primary and quasi-primary ideals in AG-groupoids. Finally, we obtain necessary and sufficient conditions of a primary ideal to be a quasi-primary ideals in AG-groupoids.

2. BASIC RESULTS

In this section we refer to [7, 8] for some elementary aspects and quote few definitions and examples which are essential to step up this study. For more details we refer to the papers in the references.

Lemma 2.1. If S is an AG-groupoid with left identity, then every right ideal is an ideal.
Proof See [8].

Lemma 2.2. If A is a left ideal of an AG-groupoid S with left identity, then aA is a left ideal in S, where $a \in S.$
Proof See [8].

Lemma 2.3. If A is a right ideal of an AG-groupoid S with left identity, then A^2 is an ideal in $S.$
Proof See [8].

Lemma 2.4. An ideal A of an AG-groupoid S is prime if and only if it is semiprime and strongly irreducible.
Proof See [8].

Lemma 2.5. A subset of an AG-3-band is a right ideal if and only if it is left.
Proof See [7].

Lemma 2.6. If S is an AG-groupoid with left identity, then a left ideal P of S is quasi prime if and only if $a(Sb) \subseteq P$ implies that either $a \in P$ or $b \in P$, where $a, b \in S$.
Proof See [8].

Lemma 2.7. If A is a proper left (right) ideal of an AG-groupoid S with left identity, then $e \notin A.e$
Proof See [8].

3. IDEALS IN AG-GROUPOIDS

The results of the following lemmas seem to play an important roleto study AG-groupoids; these facts will be used frequently and normally we shall make no reference to this lemma.

Lemma 3.1. Let S be an AG-groupoid with left identity, and let B be a left ideal of $S.$ Then $AB = \{ab : a \in A, b \in B\}$ is a left ideal in S, where $\varnothing \neq A \subseteq S.$

Proof. Suppose that S is an AG-groupoid with left identity. Let B be a left ideal of $S.$ Then $S(AB) = A(SB) \subseteq AB.$ By Definition of left ideal, we get AB is a left ideal in $S.$

Lemma 3.2. Let S be an AG-groupoid with left identity and let $a \in S.$ Then a^2S is an ideal in $S.$

Proof. By Lemma 2.2, we have a^2S is a left ideal of $S.$ Now consider

$$\begin{aligned}
(a^2r)s &= ((aa)r)s \\
&= ((ra)a)s \\
&= [e((ra)a)]s \\
&= [s((ra)a)]e \\
&= [(ra)(sa)]e \\
&= [((sa)a)r]e \\
&= [((aa)s)r]e
\end{aligned}$$

$$= [(rs)(aa)]e$$
$$= [ea^2](rs)$$
$$= a^2(rs) \in a^2S$$

for all $r, s \in S$. Therefore a^2S is an ideal in S.

Lemma 3.3. Let S be an AG-groupoid with left identity, and let A, B be left ideals of S. Then $(A:B)$ is a left ideal in S, where $(A:B) = \{r \in S : Br \subseteq A\}$.

Proof. Suppose that S is an AG-groupoid. Let $s \in S$ and let $a \in (A:B)$. Then $Ba \subseteq A$ so that

$$B(sa) \subseteq s(Ba) \subseteq sA \subseteq A.$$

Therefore $sa \in (A:B)$ so that $S(A:B) \subseteq (A:B)$. Hence $(A:B)$ is a left ideal in S.

Lemma 3.4. Let S be an AG-groupoid with left identity, and let A be a left ideal of S. Then $(A:r)$ is a left ideal in S, where $(A:r) = \{a \in S : ra \in A\}$ and $r \in S$.

Proof. By Lemma 3.3, we have $(A:r)$ is a left ideal in S.

Corollary 3.5. Let S be an AG-3-band with left identity, and let A be a left ideal of S. Then $(A:r)$ is an ideal in S, where $r \in S$.

Proof. By Lemma 3.4, we have $(A:r)$ is a left ideal in S. By Lemma 2.5, it follows that $(A:r)$ is a right ideal in S. By Lemma 2.1, we have $(A:r)$ is an ideal in S.

Remark 1. Let S be an AG-groupoid and let A be a left ideal of S. It is easy to verify that $A \subseteq (A:r)$.

2. Let S be an AG-groupoid with left identity e, and let A be a proper left (right) ideal of S. By Lemma 2.7, we have $e \notin (A:r)$, where $r \in S - A$.

3. Let S be an AG-groupoid and let A, B, C be left ideals of S. It is easy to verify that $(A:C) \subseteq (A:B)$, where $B \subseteq C$.

Corollary 3.6. Let S be an AG-3-band with left identity, and let A, B be left ideals of S. Then $(A:B)$ is an ideal in S.

Proof. This follows from Corollary 3.5.

4. PRIMARY AND QUASI-PRIMARY IDEALS IN AG-GROUPOIDS

We start with the following theorem that gives a relation between primary and quasi-primary ideal in AG-groupoid. Our starting points are the following definitions:

Definition 4.1. An ideal P is called primary if $AB \subseteq P$ implies that $A \subseteq P$ or $(((BB)B)\ldots)B = B^n \subseteq P,$ for some positive integer n, where A and B are two ideals of S.

Definition 4.2. A left ideal P is called quasi-primary if $AB \subseteq P$ implies that $A \subseteq P$ or $(((BB)B)\ldots)B = B^n \subseteq P,$ for some positive integer n, where A and B are two left ideals of S.

Remark. It is easy to see that every quasi-primary ideal is primary.

Lemma 4.3. If S is an AG-groupoid with left identity, then a left ideal P of S is quasi-primary if and only if $a(Sb)\subseteq P$ implies that $a\in P$ or $(((bb)b)\ldots)b=b^n\in P,$ for some positive integer n, $a,b\in S.$

Proof. Let P be a quasi-primary left ideal of an AG-groupoid S with left identity. Now suppose that $a(Sb)\subseteq P.$ Then by Definition of left ideal, we get $S(a(Sb))\subseteq SP\subseteq P$ that is,

$$\begin{aligned}
S(a(Sb)) &= (SS)(a(Sb)) \\
&= (Sa)(S(Sb)) \\
&= (Sa)((SS)(Sb)) \\
&= (Sa)((bS)(SS)) \\
&= (Sa)((bS)(S)) \\
&= (Sa)((SS)b) \\
&= (Sa)(Sb).
\end{aligned}$$

Since $S(a(Sb))\subseteq P$ and $S(a(Sb))=(Sa)(Sb),$ we have $(Sa)(Sb)\subseteq P$ so that $a=ea\in(Sa)\subseteq P$ or $b^n=(eb)^n\in(Sb)^n\subseteq P,$ for some positive integer n. Conversely, assume that if $a(Sb)\subseteq P$ implies that $a\in P$ or $b^n\in P$ for some positive integer n, where $a,b\in S.$ Suppose that $AB\subseteq P$, where A and B are left ideals of S such that $A\not\subset P.$ Then there exists $x\in A$ such that $x\notin P$. Now

$$x(Sy)\subseteq A(SB)\subseteq AB\subseteq P,$$

for all $y\in B$. So by hypothesis, $y^n\in P$ for all $y\in B$ implies that $B^n\subseteq P$. Hence P is quasi-primary ideal in $S.$

Lemma 4.4. If S is an AG-groupoid with left identity, then a left ideal P of S is quasi-primary if and only if $(Sa)(Sb)\subseteq P$ implies that $a\in P$ or $b^n\in P$ for some positive integer n, where $a,b\in S.$

Proof. Let P be a quasi-primary ideal of an AG-groupoid S with left identity. Now suppose that $(Sa)(Sb)\subseteq P.$ Then by Definition of left ideal, we get

$$\begin{aligned}
(Sa)(Sb) &= (SS)(ab) \\
&= S(ab) \\
&= a(Sb)
\end{aligned}$$

that is $a(Sb)=(Sa)(Sb)\subseteq P.$ By Lemma 4.3, we have $a\in P$ or $b^n\in P$ for some positive integer $n.$ Conversely, assume that if $(Sa)(Sb)\subseteq P$, then $a\in P$ or $b^n\in P$ for some positive integer n, where and $a,b\in S.$ Let $a(Sb)\subseteq P.$ Now consider

$$a(Sb)=(Sa)(Sb)\subseteq P.$$

By using given assumption, if $a(Sb)\subseteq P$, then $a\in P$ or $b^n\in P$ for some positive integer $n.$ Then by Lemma 4.3, we have P is a quasi-primary ideal in $S.$

Theorem 4.5. If S is an AG-groupoid with left identity, then a left ideal P of S is quasi-primary if and only if $ab\in P$ implies that $a\in P$ or $b^n\in P$ for some positive integer $n,$ where $a,b\in S.$

Proof. Let P be a left ideal of an AG-groupoid S with left identity. Now suppose that $ab\in P.$ Then by Definition of left ideal, we get

$$\begin{aligned}(Sa)(Sb) &= (SS)(ab)\\ &= S(ab)\\ &\subseteq SP\\ &\subseteq P.\end{aligned}$$

By Lemma 4.4, we have $a\in P$ or $b^n\in P$ for some positive integer n. Conversely, the proof is easy.

Theorem 4.6. Let S be an AG-groupoid, and let A be a quasi-primary ideal of $S.$ Then $(A:r)$ is a quasi-primary ideal in S, where $r\in S.$

Proof. Assume that A is a quasi-primary ideal of $S.$ By Lemma 3.4, we have $(A:r)$ is a left ideal in $S.$ Let $ab\in(A:r).$ Suppose that $b^n\notin(A:r)$, for all positive integer $n.$ Since $ab\in(A:r)$, we have $r(ab)\in A$ so that $a(rb)\in A.$ By Theorem 4.5, we have $a\in A\subseteq(A:r)$ or $(rb)^n\in A$, for some positive integer $n.$ Therefore $a\in(A:r)$ and hence $(A:r)$ is a quasi-primary ideal in $S.$

Theorem 4.7. Let S be an AG-groupoid with left identity and let P be a primary ideal of $S.$ If $(Sa^2)(Sb^2)\subseteq P$, then $a^2\in P$ or $b^n\in P$, for some positive integer $n,$ where $a,b\in S.$

Proof. Let P be a primary ideal of an AG-groupoid S with left identity. Suppose that $b^n\notin P$, for all positive integer n. Now assume that $(Sa^2)(Sb^2)\subseteq P$. Then by Definition of ideal, we get

$$(Sa^2)(Sb^2) = ((Sb^2)a^2)S$$

$$\begin{aligned} &= ((a^2b^2)S)S \\ &= (SS)(a^2b^2) \\ &= a^2((SS)b^2) \\ &= a^2((b^2S)S) \\ &= (b^2S)(a^2S) \end{aligned}$$

that is $(b^2S)(a^2S) \subseteq P$. By Lemma 3.2, we have a^2S and b^2S are ideals in S so that

$$\begin{aligned} a^2 &= aa \\ &= (ea)a \\ &= (aa)e \\ &= (aa)e \\ &= a^2e \in a^2S \subseteq P \end{aligned}$$

or

$$\begin{aligned} b^2 &= bb \\ &= (eb)b \\ &= (bb)e \\ &= (bb)e \\ &= b^2e \in b^2S \subseteq P \end{aligned}$$

for all $\chi \in \Gamma$. Therefore $a^2 \in P$.

Theorem 4.8. Let S be an AG-groupoid with left identity, and let P be a primary ideal of S. If $b^2a^2 \in P$, then $a^2 \in P$ or $b^n \in P$,for some positive integer n.

Proof. Let P be a primary ideal of an AG-groupoid S with left identity. Suppose that $b^n \notin P$, for all positive integer n. Now assume that $b^2a^2 \in P$. Then by Definition of ideal, we get

$$\begin{aligned} (a^2S)(b^2S) &= b^2((a^2S)S) \\ &= b^2((SS)a^2) \\ &= (SS)(b^2a^2) \\ &= S(b^2a^2) \\ &\subseteq SP \end{aligned}$$

$$\subseteq \quad P$$

that is $(a^2S)(b^2S) \subseteq P$. It is easy to see that $a^2 \in P$.

Theorem 49. Let S be an AG-3-band with left identity. Then P is a quasi -primary ideal in S if and only if S is a primary ideal in S.

Proof. The proof is straightforward.

CONCLUSIONS

Many new classes of AG-groupoids have been discovered recently. All these have attracted researchers of the field to investigate these newly discovered classes in detail. This article investigates the primary and quasi-primary ideals in AG-groupoids. Some characterizations of primary ideal and quasi-primary ideals are obtained. Moreover, we investigate the relationships between primary and quasi-primary ideals in AG-groupoids. Finally, we obtain necessary and sufficient conditions of a primary ideal to be a quasi-primary ideal in AG-groupoids.

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