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Recurrence Relations for the Moments of Order Statistics from A Generalized Beta Distribution

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ABSTRACT--- ***In this article, recurrence relations for the single and product moments of the order statistics from a generalized beta distribution are given. The use of these relations allows us to compute all the means, variances and covariances of order statistics from generalized beta distribution for*** $\gamma = 2$.

Keywords--- Distribution function, generalized beta distribution, moment, recurrence relation, order statistics.

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1. INTRODUCTION

Recurrence relations for the moments of order statistics are given by Arnold et al. 1992 [1] and Malik et al. 1988 [2]. Balakrishnan et al. 1988 [3] have reviewed many recurrence relations and identities for the moments of order statistics arising from several specific continuous distributions such as normal, gamma, exponential. Also Thomas and Samuel 2008 [4] obtained recurrence relations for the moments of order statistics from a beta distribution.

Beta distribution has a more flexible structure other than distribution functions. So, it gives the advantage of being used as more practical in the real applications. As well as, some generalizations are made for increase of this flexibility. These generalizations, use of gauss hipergeometric function very important. In this article, recurrence relations for the single and product moments of order statistics from a generalized beta distribution are given.

Let $X_1, X_2, ..., X_n$ be IID random variables from a population with cumulative distribution function $F(x)$ and probability density function $f(x)$. $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ be the order statistics of a random sample size n drawn from a generalized beta distribution. Then the k *th* single moment of $X_{r:n}$ for $1 \le r \le n$ is given by

$$\mu_{r:n}^{(k)} = r\binom{n}{r}\int_0^1 x^k [F(x)]^{r-1}[1-F(x)]^{n-r} f(x)dx \qquad (1)$$

Also, the product moment between $X_{r:n}$ and $X_{s:n}$ for $n \ge 2$ and $1 \le r < s \le n$, denoted by $\mu_{r,s:n}$, is given by

$$\mu_{r,s:n} = C_{r,s:n}\int_{-\infty}^{\infty}\int_{-\infty}^{y} xy\,[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} f(x)f(y)dxdy \qquad (2)$$

where $C_{r,s:n} = \dfrac{n!}{(r-1)!(s-r-1)!(n-s)!}$ [4].

2. GENERALIZED BETA DISTRIBUTION

Let $(z)_k$

$$(z)_k = z(z+1)(z+2)...(z+k-1)$$

Gauss hipergometric function,

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k x^k}{(c)_k k!}$$

where $(0)_k = 1$. Then, distribution function of generalized beta distributions (*cdf*),

$$F(x) = \frac{bB(a,b)}{(a+b)B(a,b+\gamma)} x^{a+b} {}_2F_1(1-\gamma, a; a+b+1; x)$$

and probability density function (*pdf*)

$$f(x) = \frac{bB(a,b)}{B(a,b+\gamma)} x^{a+b-1} {}_2F_1(1-\gamma, a; a+b; x)$$

where $0 < x < 1; a > 0; b > 0; \gamma > 0$ may be Nadarajah and Kotz 2003 [5]. Then, *cdf* and *pdf* of generalized beta distribution, for $\gamma = 2$, respectively,

$$F(x) = \left(\frac{a+b+1}{b+1}\right)\left(\frac{a+b+1}{a}\right)^{a+b} x(1-x)^{a+b} \tag{3}$$

$$f(x) = \frac{(a+b+1)(a+b)}{(b+1)}\left(\frac{a+b}{a}\right)^{a+b-1} x(1-x)^{a+b-1} \tag{4}$$

3. RECURRENCE RELATIONS FOR SINGLE MOMENTS

Using (1) and (3), we obtain the following recurrence relations for single moments of the kth from generalized beta distribution.

Theorem 1. Let $\gamma = 2$, $n \geq 2$ and $k \geq 1$ be integers. If a and b positive integers, then

$$\mu_{1:n}^{(k)} = \frac{n}{n-1}\{\mu_{1:n-1}^{(k)} - \left(\frac{a+b+1}{b+1}\right)\left(\frac{a+b+1}{a}\right)^{a+b} \sum_{i=0}^{a+b} \binom{a+b}{i} (-1)^i \mu_{1:n-1}^{(k+1+i)}\} \tag{5}$$

Proof. From (1) and (3), we can write

$$\mu_{1:n}^{(k)} = n\int_0^1 x^k [1-F(x)]^{n-2} f(x)dx$$

$$-n\int_0^1 x^k \left\{\left(\frac{a+b+1}{b+1}\right)\left(\frac{a+b+1}{a}\right)^{a+b} x(1-x)^{a+b}\right\}[1-F(x)]^{n-2} f(x)dx$$

If we expand $(1-x)^{a+b}$ binomially and simplify the resulting expression, then we obtain the results (5).

Theorem 2. Let $n \geq 2$, $2 \leq r \leq n$ and $\gamma = 2$

$$\mu_{r:n}^{(k)} = \frac{n}{r-1}\left(\frac{a+b+1}{b+1}\right)\left(\frac{a+b+1}{a}\right)^{a+b} \sum_{i=0}^{a+b}\binom{a+b}{i}(-1)^i \mu_{r-1:n-1}^{(k+1+i)} \tag{6}$$

Proof. From (1) and (3),

$$\mu_{r:n}^{(k)} = r\binom{n}{r}\int_0^1 x^k \left\{\left(\frac{a+b+1}{b+1}\right)\left(\frac{a+b+1}{a}\right)^{a+b} x(1-x)^{a+b}\right\}[F(x)]^{r-2}[1-F(x)]^{n-r} f(x)dx$$

If we expand $(1-x)^{a+b}$ binomially and simplify the resulting expression, then we obtain the results (6).

4. RECURRENCE RELATIONS FOR PRODUCT MOMENTS

In the following theorem we obtain an explicit expression for the product moment of generalized beta order statistics in terms of their single moments.

Theorem 3. Let $n \geq 2$ and $\gamma = 2$

$$\mu_{n-1,n:n} = n\left\{\mu_{1:1}\mu_{n-1:n-1} - \frac{(a+b+1)(a+b)}{(b+1)}\left(\frac{a+b}{a}\right)^{a+b+1} \sum_{i=0}^{a+b-1}\binom{a+b-1}{i}\frac{(-1)^i}{(i+3)}\mu_{n-1:n-1}^{(i+4)}\right\} \tag{7}$$

Proof. Let a and b be integers. For $\gamma = 2$,

$$\mu_{n-1,n:n} = n(n-1)\int_0^1 \left\{\int_x^1 yf(y)dy\right\} x\,[F(x)]^{n-2} f(x)dx \tag{8}$$

The inner integral on the right side of (8)

$$\int_x^1 yf(y)dy = \int_0^1 yf(y)dy - \int_0^x yf(y)dy$$

$$= \mu_{1:1} - \frac{(a+b+1)(a+b)}{(b+1)}\left(\frac{a+b}{a}\right)^{a+b+1} \int_0^x y^2(1-y)^{a+b-1}dy$$

$$= \mu_{1:1} - \frac{(a+b+1)(a+b)}{(b+1)} \left(\frac{a+b}{a} \right)^{a+b+1} \sum_{i=0}^{a+b-1} \binom{a+b-1}{i} \frac{(-1)^i}{(i+3)} x^{i+3} \tag{9}$$

Then, if we write equation (9) in equation (8), we obtain equation (7).

Theorem 4. Let $n \geq 2$ and $\gamma = 2$

$$\mu_{r,r+1:n}^{(k,l)} = \frac{n}{(n-r-1)} \left\{ \mu_{r,r+1:n-1}^{(k,l)} - \left(\frac{a+b+1}{b+1} \right) \left(\frac{a+b+1}{a} \right)^{a+b} \sum_{i=0}^{a+b} \binom{a+b}{i} (-1)^i \mu_{r,r+1:n-1}^{(k,l+1+i)} \right\}$$

Proof. The proof process of Theorem 4 is similar to the proof of Theorem 1.

Theorem 5. Let $n \geq 2$ and $\gamma = 2$

$$\mu_{r,s:n}^{(k,l)} = \frac{n}{(r-1)(s-r)(n-s)} \left\{ \left(\frac{a+b+1}{b+1} \right) \left(\frac{a+b+1}{a} \right)^{a+b} \sum_{i=0}^{a+b} \binom{a+b}{i} (-1)^i \mu_{r-1,s:n-1}^{(k+1+i,l)} \right\}$$

Proof. The proof process of Theorem 5 is similar to the proof of Theorem 2.

5. CONCLUSION

Finally, we conclude that if one uses Theorem 1, 2, 3, 4 and 5 in a systematic manner, all single and product moments of order statistics of a random sample of size n arising from a generalized beta distribution, provided the single and product moments of order statistics of lower sample sizes are known.

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