On the Modeling of Population Dynamics of a Housefly using Eigenvalues and Eigenvectors

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ABSTRACT---- ***We considered the population dynamics of a housefly in well defined stages. We use Leslie and Leftkovitch matrix models which considered eigenvalues and eigenvectors as the best approach in predicting the long- term stage growth of a housefly and the ways the population structure of a housefly vary over time. The results obtained showed that it allows production from eggs to adult ignored to determine the population structure. We concluded that these approach were the best in determining the long- term stage growth of a housefly when tested for growth and stable stage structures***

Keywords--- Population dynamics, Housefly, Eigenvalues and Eigenvectors, Fecundity, Leslie and Leftkovitch matrix model.

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1. INTRODUCTION

Wikipedia [17] defined eigenvalues as the change in magnitude of a vector that does not change in direction under a given linear transformation and eigenvector as a vector that is not rotated under a given linear transformation.

Interest on how population changes using eigenvalues and eigenvectors were considered by Ogbu, Apeh, Udeaya and Eze [1].

They studied "eigenvalues and eigenvectors as associated pairs that exist for an $n \times n$ matrix". They used eigenvalues and eigenvectors in the computation of the three first order systems in a third order scalar differential equation. The system of first order differential equation gave rise to matrix differential equation associated with each system of first order differential equation. The concept of the eigenvalues and eigenvectors are widely used in problems associated with population growth and population distribution Bernadelli [2] explored a beetle population that consist of three age classes and reported that population variation will increase as the number of years increases using Leslie matric model. Udaya, Eze and Onyema [3] studied the population dynamics of ***clarias rariepinus*** in a pond using Euler and Runge-kutta methods for approximating the increases in the yield of fish. The results obtained showed that it allowed a choice of optimal regimes of aeration, feeding and fertilization of fish for different climactic conditions in order to maximize the yield. They concluded that these approaches were the best in determining maximum yield of fish in a concrete pond when tested for growth, stocking densities and harvesting processes.

Malthus [4] suggested that the expansion of resources led to an increase in population growth reflecting the natural result of the "passion between the sexes". He also said that the population increase should be kept down to the level at which it could be supported by the operation of various checks on population growth. He categorized the checks as preventive and positive checks. Eyo [5] investigated the acceptability diet, growth performance and cost-benefit analysis for ***clarias gariepinus*** fed on varied diets enriched with dietary lipid from plant and animal sources. The result indicated that the enrichment of ***clarias rariepinus*** fingerlings diet with lipid enhances the acceptability of diet.

Caswell [6] investigated the long term behavior of Honu population having five life stages of varying lengths which are eggs (hatchings), Juveniles, sub adults, novice breeders and mature breeders.

The result indicated that the Honu population will eventually become extinct since each of the eigenvalues has a magnitude less than one. Caswell [6] suggested that size or the stage of individual is a better indicator of their chance to survive, grow and reproduce. The suitable matrix population model is stage - classified. In the case of Leftkovitch matrices, recruitment probabilities appear on the sub-diagonal and survival probabilities while for the case of Leslie matrices survival probabilities appear on the sub diagonal and there are no real recruitment probabilities. Mackean [7] described the life cycle of the housefly opining that housefly existed since the sixteen century and probably long before that, houseflies have been thought to transmit diseases.

- In the nineteenth century, experimental proof of this was put forward and the list of diseases which can be carried by house- flies has been growing ever since. For this reason and the fact that the housefly is an efficient and successful insect, it is important for use to make a close study of its life history and its habits. Cochran and Ellner [8] observed that although age is not explicit in a Leftkovitch matrix, the way such a model is built enables one to assess ages in a stage – classified population. This is because a transition matrix is computed for a given time interval (often a year). Watkinson and White [9] opined that the two models (Leftkovitch and Leslie) can only be used for insect and not for plant dynamics. This is because plant is a modular organism and are generally much more plastic in their growth than animals. Due to this plasticity, plant matrix model are usually stage or size classified as opposed to age-classified models. Von Bertalanffy [10] investigated that population dynamics of elephant which varies with respect to shoulder-height (h) and the age (x). He built software called Simulele using Matlab which he used to stimulate the growth of a population of elephant knowing the initial state. This simulele also calculated number of calves that have to be removed so that the population is stationary.

He concluded that the population dynamics of elephant increase using exponential model. Ehrlen [II] included historical effects in a matrix model that is the vital rates of individual did not depend only on their current stage but also on the stages they occupied formerly. Such historical effects seem to be frequent and it was found that they lead to biases in the asymptotic growth rates and stable stage distribution. Pencuick [12] said that Leslie matrix model cannot incorporate density feed back and take into account other environmental condition such as the rainfall. This makes Leslie matrix less deterministic by developing age and state model to annual the deficiency of Leslie matrix model.

The objective of this paper are to analyze a deterministic matrix model for the population dynamics of a housefly using eignvalues and eigenvectors.

2. PRELIMINARIES

Population Dynamics

This is study of reasons for changes in population size. John Maynard Smith [13] defined population dynamics as branch of life sciences that studies short term and long- term changes in size and age composition of population biological environmental processes influencing those changes. It deals with the way populations are affected by birth and death rate and by emigration.

Housefly

Mackean [7] defined housefly as insect common in the house that transmit diseases like diarrhea, eye inflammation and possibly tuberculosis.

Eigenvalues and Eigenvectros

Ogbu et al. [1]. Let X be a vector space such that T: X→ X

A scalar λ is said to be an eigenvalue of T if some vector $x \in$ X,

T(x) = λx. the vector $x \in$ X is called the eigenvector corresponding to the eigenvalue λ.

Fecundity-Verma [14] defined fecundity as the potential capacity of an organism to produce reproductive units such as eggs or asexual structure.

Modeling- Gerda [15] defined modeling as the representation of a real world object or system in a mathematical framework. Mathematical modeling is the use of mathematics to described real world phenomena and investigate important questions about the observed world.

Leslie Theorem and Proof

Perron–frobenius [16] the general solution of an equation such as $N_{f(t)} = L^t . N_{f(0)}$ with L diagonalizable is $n_{f(t)} = \sum C_{ij} \lambda_j^t$ where λ_j are the eignvalues of L and C_{ij} are constants whose values depend on the initial population vector .

Proof:

$N_{f(t)} = L^t . N_{f(0)}$

$= (P\Delta P^{-1})^t . N_{f(0)}$ where P is the eignvector matrix and Δ is the diagonal matrix containing the eigenvalues of L denoted by λ_i

$$N_{f(t)} = (P\Delta P^{-1})(P\Delta P^{-1}) \ldots (P\Delta P^{-1}) \ . N_{f(0)}$$
$$= P(\Delta P^{-1}P)(\Delta P^{-1}P) \ldots (\Delta P^{-1}P)\Delta P^{-1} \ . N_{f(0)}$$
$$= (P\Delta^t P^{-1}) \ . N_{f(0)}$$

Where

$$\Delta^t = \begin{bmatrix} \lambda_1^t & 0 & . & . & 0 \\ 0 & \lambda_2^t & . & . & 0 \\ 0 & 0 & . & . & 0 \\ 0 & 0 & . & . & \lambda_k^t \end{bmatrix} \qquad \text{(Perron} - \text{Frobenius) [16]}$$

Now, consider any element of $N_{f(t)}$, noticing that it is sum of the various λ_i^t, each multiplied by a value reflecting a combination of the element of P, P^{-1} and $N_{f(0)}$. That is $n_{fi(t)} = \sum C_{ij}\lambda_j^t$ *where* λ_j^t are the eigenvalues of L *and* C_{ij} are scalar constants whose values depend on the population vector.

3. MODEL FORMULATION

1. Leslie matrix model:

This model is based on age specific survival and fecundity rate. Assuming there are female members of a population of a housefly which is divided into n stages or classes. Let F_i = the fecundity of a female in the i_{th} class that is F = the average number of offspring per female in the i_{th} class.

Let P_i = the probability that female in the i_{th} class will survive to become a member of the $(i+1)^{th}$ class.

Let

$$X^{(k)} = \begin{bmatrix} x_1^{(k)} \\ x_1^{(k)} \\ \cdot \\ \vdots \\ x_1^{(k)} \end{bmatrix} = \begin{bmatrix} \text{population of stage 1 female in year k} \\ \text{population of stage 2 female in year k} \\ \cdots \\ \cdots \\ \cdots \\ \text{population of stage n female in year k} \end{bmatrix}$$

Then $x_1^{(k+1)} = F_1 x_1^{(k+1)} + F_2 x_2^{(k)} + \cdots + F_{n-1} x_{n-1}^{(k)} + F_n x_n^{(k)}$

$x_2^{(k+1)} = P_1 x_1^{(k)}$

$x_3^{(k+1)} = P_2 x_2^{(k)}$

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$x_n^{(k+1)} = P_{n-1} x_{n-1}^{(k)}$

Leslie matrix that describe the change in the population over time is given by

$$L = \begin{bmatrix} F_1 & F_2 & \cdots & F_{n-1} & F_n \\ P_1 & 0 & \cdots & 0 & 0 \\ 0 & P_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & P_{n-1} & 0 \end{bmatrix}$$

System of linear equation is given by $x^{(k+1)}=Lx^{(k)}$ for n stage, we have $x^n = L^n X^0$ where L^n denote that stages of the population with change over time.

2. Leftkovitch matrix model

This model is based on stage specific rate. Let $\vec{n}(t)$ stands for the number of individual at stage t. The coefficient matrix which can be constructed from the probabilities of transition among the various stages is given by.

$$A = \begin{pmatrix} P_1 & F_2 & F_3 & F_4 \\ G_1 & P_2 & 0 & 0 \\ 0 & G_2 & P_3 & 0 \\ 0 & 0 & G_3 & P_4 \end{pmatrix}$$

Matrix equation is given by $\vec{n}(t) = A\vec{n}(t-1)$

where

$\boldsymbol{P_i}$ = the probability of surviving and staying in stage i.

$\boldsymbol{G_i}$ = the probability of surviving and growing from stage i to stage $i+1$.

$\boldsymbol{F_i}$ = the fertility of stage i.

Stable stage distribution is given $\vec{n}(t) = A^t\vec{n}(0)$ n (t) = A^t n (0)

4. MAIN RESULT

We make the following assumption

1. Female population of housefly is divided into two, stages each one year in length.
2. Female in the first stage produce no offspring
3. Female in the first stage have 60% of surviving to the second stage.

4. Female in the second stage produce average of four female offspring per year but are guaranteed to die after one year in stage 2.
5. Total females in the first stage is about 100
6. Total females in the second stage is about 100.

Then the matrix L is given as

$$L = \begin{bmatrix} 0 & 4 \\ 0.6 & 0 \end{bmatrix}$$

Eigenvalues of L are

$$|L - \lambda I| = 0 \rightarrow \begin{vmatrix} -\lambda & 4 \\ 0.6 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 2.4 = 0$$

$$\lambda^2 = 2.4$$

$$\lambda = \pm\sqrt{2.4} = \pm 1.55$$

$$\lambda_1 = 1.55, \lambda_2 = -1.55$$

Eigenvector for λ_1

$$(L - \lambda I)X = 0 \rightarrow \begin{bmatrix} -1.55 & 4 \\ 0.6 & -1.55 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \qquad (3.2)$$

The solution for (3.2)

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C \begin{bmatrix} 2.58 \\ 1 \end{bmatrix} \quad (C = constant)$$

$e_1 = (2.58,\ 1)$ where e_1 = eigenvector for λ_1

Eigenvector for λ_2

$$(L - \lambda I)X = 0 \rightarrow \begin{bmatrix} 1.55 & 4 \\ 0.6 & 1.55 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (3.3)$$

The solution for (3.3)

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = K \begin{bmatrix} -2.58 \\ 1 \end{bmatrix} \quad (K = constant)$$

$e_2 = (-2.58,\ 1)$ where e_2 = eigenvector for λ_2

To determine the long term stage – growth population we express

$$X^{(0)} = C_1 e_1 + C_2 e_2 \quad (C_1, C_2 \ are \ constants)$$

$$(100, 100) = C_1(2.15, 1) + C_2(-2.15, 1)$$

$$(100, 100) = 2.58C_1 + C_1 = 2.58C_2 + C_2$$

$$100 = 2.58C_1 - 2.58C_2 \ \ldots(i) \qquad (3.4)$$

$$100 = C_1 - C_2 \ \ldots(ii)$$

Solving equation (10) and (II) simultaneously we have

$C_1 = 30.62$ and $C_1 = 69.38$

Long term stage – growth population becomes

$X^{(0)} = 69.38e_1 + 30.62e_2$

Using $x^{k+1} = Lx^k$

For K= 0, $x^1 = Lx^{(0)}$

$$= L(69.38e_1 + 30.62e_2)$$

$$= 69.38(Le_1) + 30.62(Le_2)$$

$$= 69.38\big(1.55(2.58, 1)\big) + 30.62(-1.55(-2.58, 1))$$

$$= (277.45,\ 107.539) + (122.45,\ -47.61)$$

$$= (399.9,\ 59.93)$$

For K= 1

$$X^2 = L^2 X^{(0)}$$

$$X^2 = L^2(69.38e_1 + 30.62e_2)$$

$$X^2 = 69.38(L^2 e_1) + 30.62(L^2 e_2)$$

$$= 69.38(2.4025\ (2.58, 1)) + 30.62(2.4025(-2.58, 1))$$

$$= (430.05, 166.69) + (-189.80, 73.56)$$

$$= (-189.80, 73.56).$$

B. We consider the four life stages of a housefly which are eggs, larva, pupa and Adult. The ages, annual survivorship and number of eggs laid per year for each stage are provided in the following table.

Stage description	Ages	Annual survivorship	Eggs laid for each stage.
Eggs	4	0.0110	0
Larva	5	0.01370	0
Pupa	9	0.0247	0
Adult	10-14	0.0329	125

The Leftkovitch matrix is

$$A = \begin{bmatrix} 0.0110 & 0 & 0 & 125 \\ 0.0027 & 0.01370 & 0 & 0 \\ 0 & 0.11 & 0.0247 & 0 \\ 0 & 0 & 0.0082 & 00329 \end{bmatrix}$$

The Eigenvalues are

$\lambda_1 = -0.0542$ $\lambda_2 = 0.0206$

$\lambda_3 = 0.0206$ $\lambda_4 = 0.0954$

The corresponding eigenvectors are

$$e_1 = \begin{bmatrix} -0.9992 \\ 0.9993 \\ 0.9993 \\ 0.9994 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0.0397 \\ 0.0034 \\ 0.0034 \\ 0.0330 \end{bmatrix} \quad e_3 = \begin{bmatrix} -0.0055 \\ -0.0054 \\ -0.0054 \\ -0.0051 \end{bmatrix} \quad e_4 = \begin{bmatrix} 0.005 \\ 0.0001 \\ 0.0001 \\ 0.0007 \end{bmatrix}$$

The dominant eigenvalue is $\lambda_4 = 0.0954$.

The eigenvector corresponding to the dominant eignvalue is

$$e_4 = \begin{bmatrix} 0.005 \\ 0.0001 \\ 0.0001 \\ 0.0007 \end{bmatrix}$$

The stable stage distribution is given by

$$U_\infty = \begin{bmatrix} 0.7143 \\ 0.1429 \\ 0.1429 \\ 0.0007 \end{bmatrix}$$

5. DISCUSSION

We discovered that eignvalues and eigenvectors play important roles in determining the long-term stage growth of a population of a housefly. When K= 0, population of $x^{(1)}$ is (399.9, 59.93) showing that there is an instinct of growth as the value of K increases.

We also discovered that Leftkovitch matrix model using eignvalues and eigenvectors is dependent on stage for its distribution. The largest eigenvalue $\lambda_4 = 0.0954$ gives the stable stage structure that is component of the eigenvector corresponding to the dominant eigenvalue which gives the proportion of the species in each stage in the long- term

The stable stage distribution U_∞ (U – infinity) given as

$$U_\infty = \begin{bmatrix} 0.7143 \\ 0.1429 \\ 0.1429 \\ 0.0007 \end{bmatrix}$$

was derived by normalizing the dominant eigenvector so that its component add up to one. This shows that there is a certainty of growth in the population of housefly based on the stages of its life cycle. The result obtained show that the models use fecundity rate for the population of housefly and transition from one stage to another to describe the way population structure varies over time.

6. CONCLUSION

Eignvalues and eigenvector of Leslie and Lefkovitch matrix modles as predicated is the best approached in determining the long term stage growth of a housefly and its stable stage structure.

7. REFERENCES

1. Ogbu, H.M, Apeh K.O, Udaya Ode and Eze E.O: the first order system of equations in a third order scalar differential equations with some agentvalues and agentvectors of some third order matrics differential equation, Devion Science company, journal of Nigeria Mathematical society Vol 7, Number 1, Dec 104-107 9 (2012).

2. Bernadelli, H'. Population waves J. Burma Res. Soc, 31:1-18 (1914).
3. Udaya, C.O. Eze, E.O. and Onyema, M.K. On the mathematical modelling of population dynamics and growth of Clarias Glarias in a pond ,10SR journal of mathematical Vol 9, pp 23-25 (2013).
4. Malthus. An essay on the principle of population in st Paul's church –yards, London, UK (1798).
5. Eyo: Management of inland capture fisheries and challenges to fish production in Nigeria (2003).
6. Caswell, H, The Dynamics of plant population ecology 81:1675-1684 (2000).
7. Caswell, H, Matric population Modes, Construction Analysis and interpretation 2nd edition, the publishers S.L (2001).
8. Mackean, D.G.; Biology Teaching Resources, the housefly. Http:www.biologyresources.com (2010).
9. Cochran and Ellner: simple method for calculating age base life history parameters for stage- structured population. (1992).
10. Watkinson and white: Some life-history conbsequences of modular contruction in plants. Lond 313:31-51 (1985).
11. Von Bertalamffy: Determination of the von Bertalanffy growth equation for three sliver hake stocks (1938).
12. Ehrlen, J: The dynamics of plant population ecology 81:1675-1684 (2000)
13. Pencuick: Modelling population Dynamic of Elephant –chez Sam. (1968).
14. John Maynard Smith: On the origin of life published on Dec. 4 2012 (2012)
15. Verma: Gene probes 1,
16. Gerda: website, www.google.com (2014)
17. Perom Frobenius: Theorem of Perom Frobeniustype for matrices pp 494-495(1914)
18. Wikipedia