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On the Classical Primary Radical of Submodules

PairoteYiarayong[1], and Phakakorn Panpho[2]

[1] Faculty of Science and Technology,
Pibulsongkram Rajabhat University,
Phitsanuloke 65000, Thailand
E-mail: pairote0027 {at} hotmail.com

[2] Faculty of Science and Technology,
Pibulsongkram Rajabhat University,
Phitsanuloke 65000, Thailand

ABSTRACT—*In this paper we characterize the classical primary radical of submodules and classical primary submodules of modules over a commutative rings with identity. Those are extended from radical classical primary, radical primary, and primary of submodules, respectively. Moreover, we investigate relationships between classical primary radical and radical primary submodules.*

Keywords— classical primary submodule, primary submodule, classical primary radical of submodule, primary radical of submodul, primary ideal.

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1. INTRODUCTION

Throughout this paper all rings are commutative with identity and all modules are unitary. we recall that a proper ideal P of R is called a *primary ideal* if $ab \in P,$ where $a,b \in R,$ implies that either $a \in P$ or $b^n \in P,$ for some positive integer $n.$ The notion of primary ideal was generalized by Fuchs (1947) through defining an ideal P of a ring R to be its *quasi-primary* if its radical is a prime ideal, i.e., if $ab \in P,$ where $a,b \in R,$ then either $a^n \in P$ or $b^n \in P,$ for some positive integer $n.$ A proper submodule N of an R-module M is a *primary submodule* of M if for $m \in M$ and $r \in R$ such that $rm \in N,$ then $m \in N$ or $r \in \sqrt{(N:M)} = \{a \in R \mid a^n M \subseteq N,$ for some positive integer $n\}$

An R-module M is a *primary module* if every proper submodule N of M is a primary submodule of $M.$ A *classical primary submodule* in M as a proper submodule N of M such that if $abK \subseteq N,$ where $a,b \in R$ and $K \leq M,$ then either $aK \subseteq N$ or $b^n K \subseteq N,$ for some positive integer $n.$ Clearly, in case $M = R,$ where R is any commutative ring, classical primary submodules coincide with primary ideals. The idea of decomposition of submodules into classical primary submodules was introduced by Baziar and Behboodi (2009). The *primary radical* of N in $M,$ denoted by $prad_M(N),$ is defined to be the intersection of all primary submodules containing $N.$ Should there be no primary submodule of M containing $N,$ then we put $prad_M(N) = M.$ Radicals have been investigated in a number of papers, for example, in ([1], [2]). A *classical primary radical* of N in $M,$ denoted by $c.prad_M(N),$ is defined to be the intersection of all classical primary submodules containing $N.$ Should there be no classical primary submodule of M containing $N,$ then we put $c.prad_M(N) = M.$

In this paper we characterize the classical primary radical of submodules and classical primary radical formula of modules over commutative ring with identity. Moreover, we investigate relationships between classical primary radical and radical primary submodules.

2. BASIC RESULTS

In this section we introduce the concept of the primary radical and give some useful properties about it.

Definition 2.1. [6] A proper ideal P of R is called a primary ideal if $ab \in P,$ where $a,b \in R,$ implies that either $a \in P$ or $b^n \in P,$ for some positive integer $n.$

Lemma 2.2. If P is a primary ideal of R and let $r^n, c^n \notin P$ for all positive integers $n,$ then $(rc)^n \notin P.$

Proof. Suppose that $r^n c^n = (rc)^n \in P,$ for some positive integers $n.$ Since P is a primary ideal of $R,$ it follows that $r^n \in P$ or $r^{ns} = (r^n)^s \in P,$ for some positive integers $s.$ This is a contradiction to the fact that $r^n, c^n \in P,$ for all positive integer $n.$ Therefore, $(rc)^n \notin P.$

Lemma 2.3. [6] Let M be an R-module and N be a proper submodule of $M.$ Then N is classical primary if and only if for every submodule K of M such that $K \subseteq N, (N : K)$ is a primary ideal of $R.$
Proof. The proof is obvious.

Lemma 2.4. [14] Let M be a Noetherian R-module and N be a proper submodule of $M.$ The following statements are equivalent:

(1) N is a classical primary submodule.

(2) For every $a,b \in R$ and $m \in M,$ $abm \in N,$ it implies that either $am \in N$ or $b^n m \in N$ for some positive integer $n.$

(3) For every $m \in M - N,$ it is a primary ideal of $R.$

Definition 2.5. [1] The primary radical of N in $M,$ denoted by $prad_M(N),$ is defined to be the intersection of all primary submodules of M containing $N.$ If there is no primary submodule of M containing $N,$ then $prad_M(N) = M.$

3. SOME BASIC PROPERTIES OF THE PRIMARY SUBMODULES

The results of the following lemmas seem to be at the heart of the theory of primary submodules; these facts will be used so frequently that normally we shall make no reference to this lemma.

Lemma 3.1. If N is a proper submodule of an R-module M and P is a proper ideal of $R,$ then $PM + N \subseteq (N,P) = \{x \in M \mid cx \in PM + N, \text{where } c \in R - P\}.$

Proof. Let $x \in PM + N.$ Then $x = pm + k,$ where $p \in P, m \in M$ and $k \in N.$ There exists $1 \in R - P$ so that

$$1x = x = pm + k \in PM + N.$$

It follows that $x \in (N,P)$ and hence $PM + N \subseteq (N,P).$

Lemma 3.2. If N is a proper submodule of an R-module M and P is a proper ideal of R, then (N,P) is a submodule of M.

Proof. It follows from Lemma 3.1 that $(N,P) \neq \varnothing$. To show that submodule properties of (N,P) hold, let $r \in R$ and $x, y \in (N,P)$. There exists c_1, c_2 with $c_1, c_2 \in R - P$ such that $c_1x, c_2y \in PM + N$. Then $c_1x = p_1m_1 + n_1$ and $c_2y = p_2m_2 + n_2$, where $p_1, p_2 \in P, m_1, m_2 \in M$ and $n_1, n_2 \in N$. Now consider

$$\begin{aligned} c_1c_2(x+y) &= c_1c_2x + c_1c_2y \\ &= c_2(c_1x) + c_1c_2y \\ &= c_2(p_1m_1 + n_1) + c_1(p_2m_2 + n_2) \\ &= c_2p_1m_1 + c_2n_1 + c_1p_2m_2 + c_1n_2 \\ &= (c_2p_1m_1 + c_1p_2m_2) + (c_2n_1 + c_1n_2) \in PM + N \end{aligned}$$

and

$$\begin{aligned} c_1rx &= r(c_1x) \\ &= r(p_1m_1 + n_1) \\ &= rp_1m_1 + rn_1 \in PM + N. \end{aligned}$$

Therefore $x + y \in (N,P)$ and $rx \in (N,P)$. Hence (N,P) is a submodule of M.

Lemma 3.3. If N is a proper submodule of an R-module M and P is a primary ideal of R, then $(N,P) = M$ or (N,P) is a primary submodule of M.

Proof. Suppose that $(N,P) \neq M$. We will show that (N,P) is a primary submodule of M . By Lemma 3.2, we have (N,P) is a submodule of M. To see the prime property of (N,P), let $r \in R$ and $m \in M$ such that $rm \in (N,P)$. We will prove that

$$m \in (N,P) \text{ or } rM \subseteq (N,P).$$

Since $rm \in (N,P)$, there exists c with $c \in R - P$, such that $crm \in PM + N$. We have 2 cases to consider; $r \in P$ and $r \notin P$.

Case 1. If $r \in P$, then by the definition of (N,P), we have $rM \subseteq PM \subseteq PM + N \subseteq (N,P)$. Therefore $rM \subseteq (N,P)$.

Case 2. If $r \notin P$, then $cr \notin P$. Thus $m \in (N,P)$.

Therefore (N,P) is a primary submodule of M.

Corollary 3.4. If N is a proper submodule of an R-module M and let P be a primary ideal of R, then $(N,P) = M$ or (N,P) is a cliassical primary submodule of M.

Proof. The proof is obvious.

4. SOME BASIC PROPERTIES OF THE CLASSICAL PRIMARY SUBMODULES

Before theorizing of the theorem of the relationship between primary radicals radical and classical primary radical submodules of submodules of an R-module M. Our starting point is the following definition:

Definition 4.1. The classical primary radical of N in M, denoted by $c.prad_M(N)$, is defined to be the intersection of all classical primary submodules of M containing N. If there is no classical primary submodule of M containing N, then $c.prad_M(N) = M$.

Lemma 4.2. If N and K are submodules of an R-module M such that $N \subseteq K$, then $c.prad_M(N) \subseteq c.prad_M(K)$.

Proof. If there is no classical primary submodule of M containing K, then $c.prad_M(K) = M$. Since $c.prad_M(N)$ is a submodule of M, we have

$$c.prad_M(N) \subseteq M = c.prad_M(K).$$

If there exists a classical primary submodules of M containing K, then $c.prad_M(K)$ is a submodule of M, with $K \subseteq c.prad_M(K)$. Since $N \subseteq K$, we have

$$N \subseteq c.prad_M(K).$$

Therefore $c.prad_M(N) \subseteq c.prad_M(K)$.

Corollary 4.3. If N is a submodule of an R-module M, then $c.prad_M(N) \subseteq c.prad_M(M) = M$.

Proof. It follows from Lemma 4.2.

Proposition 4.4. Let N and K be submodules of an R-module M. Then

(1) $N \subseteq c.prad_M(N)$,

(2) $c.prad_M(c.prad_M(N)) = c.prad_M(N)$,

(3) $c.prad_M(N \cap K) \subseteq c.prad_M(N) \cap c.prad_M(K)$,

(4) $c.prad_M(N + K) = c.prad_M(N) + c.prad_M(K)$.

Proof. (1) Obviously, $N \subseteq P$ for every classical primary submodule P.

(2) Since $N \subseteq c.prad_M(N)$, by Lemma 4.2,

$$c.prad_M(N) \subseteq c.prad_M(c.prad_M(N)).$$

We will show that $c.prad_M(c.prad_M(N)) \subseteq c.prad_M(N)$. If there is no classical primary submodule of M containing N, then $c.prad_M(N) = M$. Thus

$$c.prad_M(c.prad_M(N)) \subseteq c.prad_M(N).$$

If there exists classical primary submodules of M containing N, then there exists classical primary submodule W, with $N \subseteq W$. Then by the definition of $c.prad_M(N)$, $c.prad_M(N) \subseteq W$. It follows that

$$c.prad_M(c.prad_M(N)) \subseteq c.prad_M(N)$$

and hence $c.prad_M(c.prad_M(N)) = c.prad_M(N)$.

(3) Since $N \cap K \subseteq N$ and $N \cap K \subseteq K$, by Lemma 4.2,

$$c.prad_M(N \cap K) \subseteq c.prad_M(N)$$

and

$$c.prad_M(N \cap K) \subseteq c.prad_M(K).$$

Therefore $c.prad_M(N \cap K) \subseteq c.prad_M(N) \cap c.prad_M(K)$.

(4) We will show that $c.prad_M(N+K) \subseteq c.prad_M(N) + c.prad_M(K)$. It is clear that

$$N + K \subseteq c.prad_M(N) + c.prad_M(K),$$

so that $c.prad_M(N+K) \subseteq c.prad_M(N) + c.prad_M(K)$. On the other hand, we will show that

$$c.prad_M(N) + c.prad_M(K) \subseteq c.prad_M(N+K).$$

Since $N \subseteq N+K$ and $K \subseteq N+K$, we have

$$c.prad_M(N) \subseteq c.prad_M(N+K)$$

and

$$c.prad_M(K) \subseteq c.prad_M(N+K).$$

Thus $c.prad_M(N) + c.prad_M(K) \subseteq c.prad_M(N+K)$. Hence $c.prad_M(N+K) = c.prad_M(N) + c.prad_M(K)$.

Lemma 4.5. If N and K are submodules of an R-module M such that $N \subseteq K$, then $c.prad_K(N) \subseteq c.prad_M(N)$.

Proof. If there is no classical primary submodule of M containing N, then $c.prad_M(N) = M$. Since $c.prad_K(N)$ is a submodule of K, we have

$$c.prad_K(N) \subseteq K \subseteq M = c.prad_M(N).$$

There exists a classical primary submodules of M containing N, then $c.prad_M(N)$ is a submodule of M, with $N \subseteq c.prad_M(N)$. Let W be a classical primary submodule of M containing N. We have 2 cases to consider; $W \subseteq K$ and W Ú K.

Case 1. $W \subseteq K$. Then is a classical primary submodules of K containing N, so that $w.rad_K(N) \subseteq W$.

Case 2. W Ú K. It is clear that $K \cap W$ is a classical primary submodule of K. Since $N \subseteq K \cap W$, we have $c.prad_K(N) \subseteq K \cap W \subseteq W$.

Hence, $c.prad_K(N) \subseteq c.prad_M(N)$.

Theorem 4.6. Let N be a proper submodule of an R-module M. Then $c.prad_M(N) \subseteq \bigcap\{(N,P) \mid P$ is a primary ideal of R$\}$.

Proof. Let $B = \bigcap\{(N,P) \mid P$ is a primary ideal of R$\}$. We show that $c.prad_M(N) \subseteq B$. Let $m \in c.prad_M(N)$ and let $(N,P) \in B$. Then by Corollary 3.4, $(N,P) = M$ or (N,P) is a classical primary submodule of M.

Case 1. If $\left(N,P\right) = M$, then it is trivial that $m \in (K,P)$.

Case 2. If (N,P) is a classical primary submodule of M and $N \subseteq PM + N \subseteq (N,P)$, then $m \in (N,P)$. Therefore $c.prad_M(N) \subseteq \bigcap\{(N,P) \mid P$ is a primary ideal of R$\}$.

Lemma 4.7. Let N be a proper submodule of an R -module M. Then $prad_M(N) = \bigcap\{(N,P) \mid P$ is a primary ideal of $R\}$.

Proof. Let $B = \bigcap\{(N,P) \mid P$ is a primary ideal of $R\}$. We will show that $B \subseteq prad_M(N)$. Let L be a primary submodule of M, containing N and let $m \in B$. Since L is a primary submodule of M, we have $(L:M)$ is primary ideal of R. There exists c with $c \in R-(L:M)$ such that $cm \in (L:M)M+N$. Therefore $cm = hs+k$, where $h \in (L:M)$, $s \in M$ and $k \in N$. Since $hM \subseteq L$ and $N \subseteq L$, we have $hs \in L$ and $k \in L$. Then $cm \in L$. Since cM Ú L and L is a primary submodule of M, we have $m \in L$. It follows $m \in prad_M(N)$ and hence $B \subseteq prad_M(N)$. Next, we will show that $prad_M(N) \subseteq B$. Let $m \in prad_M(N)$ and let $(N,P) \in B$. Then by Lemma 3.3 $(N,P) = M$ or (N,P) is a primary submodule of M.

Case 1. If $(N,P) = M$, then it is trivial that $m \in (N,P)$.

Case 2. If (N,P) is a primary submodule of M and $N \subseteq PM+N \subseteq (N,P)$, then $m \in (N,P)$. Therefore $B \subseteq prad_M(N)$ and hence $prad_M(N) = \bigcap\{(N,P) \mid P$ is a primary ideal of R$\}$.

Theorem 4.8. Let N be a proper submodule of an R -module M. Then $c.prad_M(N) \subseteq prad_M(N)$.

Proof. It follows from Theorem 4.6 and Lemma 4.7.

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