Modified Goodness of Fit Tests for Rayleigh Distribution

Vanida Pongsakchat[1] and Pattaraporn Tonhaseng[2]

[1] Department of Mathematic, Faculty of Science, Burapha University
Chonburi, Thailand
Email: vanida [AT] buu.ac.th

[2] Department of Mathematic, Faculty of Science, Burapha University
Chonburi, Thailand
Email: pattaraporntonhaseng [AT] gmail.com

ABSTRACT— ***The modified goodness of fit tests for the Rayleigh distribution are studied. The critical values of modified Kolmogorov-Smirnov, Cramer-von-Mises and Anderson-Darling tests are obtained by Monte Carlo simulation for different sample sizes and significant levels. The type I error rate and power of these tests are studied and compared. The results show that all of the three tests have type I error rate close to the significant levels. Under several alternative distributions, it is founded that when the sample size is large, modified Anderson-Darling has the largest power in all cases. However, when the sample size is small, skewness of the distribution plays an important role. For the more skewed distribution, the modified Anderson-Darling test has more power than the others, while the modified Cramer-von-Mises has the largest power when the distribution is less skewed.***

Keywords— Goodness-of-fit test, Rayleigh distribution, Monte Carlo simulation, Power of the test

1. INTRODUCTION

The Rayleigh distribution was introduced by Lord Rayleigh in 1880 [8]. It has been used in many fields such as in medical research, estimate the noise variance in an MRI image from background data, in physics, model processes such as wave heights, sound and light radiation, radio signals, wind power and ultrasound image modeling etc. The Rayleigh distribution is also used in the field of reliability theory and survival analysis [2][4].

If X is a Rayleigh random variable, from Johnson, Kotz & Balakrishnan [5] the probability density function and the cumulative distribution function with one parameter (σ) are given by

$$f(x) = \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)}, \qquad 0 \le x < \infty, \sigma > 0 \tag{1}$$

and

$$F(x) = 1 - e^{-x^2/2\sigma^2}, \quad 0 \le x < \infty, \sigma > 0 \tag{2}$$

Often, it is importance to know whether the data come from the certain distribution. The statistical test called goodness of fit can be used for this purpose. The well-known goodness of fit tests are Kolmogorov-Smirnov (KS), Cramer-von-Mises (CvM) and Anderson-Darling (AD) tests. These tests are based on the empirical cumulative distribution. However, in practice, the parameters of the hypothesized distribution need to be estimated from the data. In this case, the standard critical values tables of these tests are no longer valid. The Kolmogorov-Smirnov, Cramer-von-Mises and Anderson-Darling tests are called the modified tests when the parameters of the hypothesize distribution must be estimated. Many researchers proposed the table of critical values of the modified tests for some distributions using Monte Carlo techniques.

Lilliefors [6,7] obtained the critical values tables for a modified Kolmogorov-Smirnov test for the normal and exponential distributions. Among the many authors that haves constructed the tables of critical values for various modified goodness of fit tests for different types of distribution. Further details can be seen in [1,3,9,10,11].

In this paper, critical values for modified KS, modified CvM and modified AD using Monte Carlo techniques are obtained for Rayleigh distribution with unknown parameter. Tables of critical values for various sample and significant levels are provided. The type I error rate and power of these tests are compared and discussed.

2. MODIFIED GOODNESS-OF-FIT TESTS

A single random sample of size n is drawn from a population with unknown cumulative distribution function $F_n(x)$ and we wish to test the hypotheses

$$H_0 : F_n(x) = F_0(x), \quad \text{for all } x$$

$$H_a : F_n(x) \neq F_0(x), \quad \text{for some } x$$

where $F_0(x)$ is hypothesized cumulative distribution function.

In this section, we introduce the modified KS, CvM and AD tests for Rayleigh distribution when the parameter are estimated. The estimator are used in $F_0(x)$ for modified goodness of fit tests.

For Rayleigh distribution, parameter σ can be estimated using maximum likelihood estimation. From [5] the maximum likelihood estimator of σ is

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n} x_i^2}{2n}} \tag{3}$$

1. Modified Kolmogorov-Smirnov statistic

$$\begin{aligned} D_n^+ &= \max_{1\le i\le n}\left\{\frac{i}{n} - F_0(x_i;\hat{\sigma})\right\}; \\ D_n^- &= \max_{1\le i\le n}\left\{F_0(x_i;\hat{\sigma}) - \frac{i-1}{n}\right\}; \\ D_n &= \max\left\{D_n^+, D_n^-\right\}. \end{aligned} \tag{4}$$

2. Modified Cramer-von-Mises statistic

$$W_n^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left(F_0(\mathrm{x}_i;\hat{\sigma}) - \frac{2i-1}{2n}\right)^2. \tag{5}$$

3. Modified Anderson-Darling statistic

$$A_n^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left(\log\left\{F_0(\mathrm{x}_i;\hat{\sigma})\right\} + \log\left\{1 - F_0(x_{n-i+1};\hat{\sigma}\right\}\right). \tag{6}$$

3. CRITICAL VALUES TABLES

For each of the three tests (KS, CvM, AD), each sample size $n = 5(5)30(10)50$ and 100, a random sample $x_1, x_2, ..., x_n$ is generated from Rayleigh distribution. The random sample is used to estimate the MLE estimator σ and then used to determine $F_0(x)$. The test statistics are calculated and recorded for a given values of n. The process is repeated 10,000 times. The 10,000 number of statistics values are arranged in ascending order and the 80%, 85%, 90%, 95% and 99% quantiles are founded thus establishing the critical value for the particular test and sample size. Table 1-3 show critical values for modified KS, CvM and AD tests.

Based on Table 1-3, critical values of modified KS, CvM and AD test increase as the significant level (α) decrease. For modified CvM and AD tests, the critical values increase as the sample size increases while the critical values for modified KS decrease as sample size increases. Moreover, for the specific sample size the critical values for modified AD test is the biggest for all significant levels.

Table 1: Critical values for modified KS test

n	α = 0.20	α =0.15	α =0.10	α =0.05	α =0.01
5	0.3604	0.3769	0.4045	0.4420	0.5122
10	0.2626	0.2768	0.2955	0.3247	0.3814
15	0.2171	0.2289	0.2444	0.2687	0.3172
20	0.1893	0.1996	0.2133	0.2346	0.2770
25	0.1701	0.1783	0.1917	0.2106	0.2492
30	0.1560	0.1645	0.1757	0.1932	0.2283
40	0.1357	0.1431	0.1528	0.1680	0.1989
50	0.1217	0.1284	0.1360	0.1508	0.1787
100	0.0869	0.0916	0.0978	0.1075	0.1274

Table 2: Critical values for modified CvM test

n	α = 0.20	α =0.15	α =0.10	α =0.05	α =0.01
5	0.1293	0.1463	0.1704	0.2102	0.2987
10	0.1295	0.1473	0.1727	0.2164	0.3204
15	0.1297	0.1476	0.1734	0.2183	0.3263
20	0.1297	0.1478	0.1736	0.2193	0.3304
25	0.1297	0.1478	0.1737	0.2194	0.3297
30	0.1300	0.1479	0.1740	0.2201	0.3316
40	0.1296	0.1477	0.1739	0.2202	0.3333
50	0.1296	0.1477	0.1740	0.2207	0.3343
100	0.1300	0.1482	0.1746	0.2212	0.3363

Table 3: Critical values for modified AD test

n	α = 0.20	α =0.15	α =0.10	α =0.05	α =0.01
5	0.7622	0.8533	0.9809	1.1751	1.6975
10	0.7884	0.8848	1.0239	1.2683	1.8990
15	0.7971	0.8956	1.0376	1.2863	1.9156
20	0.8013	0.9002	1.0431	1.2965	1.9319
25	0.8040	0.9036	1.0468	1.2996	1.9312
30	0.8063	0.9064	1.0501	1.3057	1.9378
40	0.8071	0.9078	1.0521	1.3084	1.9445
50	0.8076	0.9081	1.0542	1.3101	1.9506
100	0.8128	0.9133	1.0583	1.3163	1.9538

4. TYPE I ERROR RATE

Type I error rate of the modified KS, CvM and AD test are obtained for sample size $n = 10(10)50$ and 100 selected from Rayleigh distribution with σ =0.1, 1.2 2.8 and 5.5. For each selected test, sample size, and hypothesize distribution, 10,000 random samples are generated and the tests are conducted using critical values at significant level α = 0.05 and 0.01 in this paper. The proportion of rejections are recorded as the type I error rate and reported in Table 4-5.

From Table 4 and 5, it is founded that type I error rate of the three modified tests are close to the significant levels for all sample sizes and σ . Hence, these tests can control probability of type I error.

Table 4 Type I error rate for modified KS, CvM and AD test (α =0.05)

n	Test	σ = 0.1	σ = 1.2	σ = 2.8	σ = 5.5
10	KS	0.0502	0.0481	0.0520	0.0466
	CvM	0.0542	0.0486	0.0527	0.0464
	AD	0.0540	0.0486	0.0513	0.0482
20	KS	0.0523	0.0479	0.0471	0.0487
	CvM	0.0488	0.0488	0.0480	0.0487
	AD	0.0482	0.0494	0.0478	0.0463
30	KS	0.0478	0.0500	0.0516	0.0463
	CvM	0.0462	0.0504	0.0494	0.0466
	AD	0.0486	0.0491	0.0507	0.0469
40	KS	0.0501	0.0528	0.0511	0.0478
	CvM	0.0487	0.0522	0.0492	0.0503
	AD	0.0488	0.0492	0.0491	0.0484
50	KS	0.0535	0.0515	0.0480	0.0491
	CvM	0.0507	0.052	0.0475	0.0488
	AD	0.0513	0.0514	0.0481	0.0502
100	KS	0.0526	0.0516	0.0506	0.0488
	CvM	0.0494	0.0478	0.0511	0.0492
	AD	0.0480	0.0503	0.0492	0.0477

Table 5 Type I error rate for modified KS, CvM and AD test (α =0.01)

n	Test	σ = 0.1	σ = 1.2	σ = 2.8	σ = 5.5
10	KS	0.0123	0.0105	0.0100	0.0101
	CvM	0.0115	0.0101	0.0102	0.0092
	AD	0.0103	0.0097	0.0109	0.0106
20	KS	0.0105	0.0104	0.0103	0.0075
	CvM	0.0106	0.0098	0.0088	0.0077
	AD	0.0100	0.0102	0.0093	0.0078
30	KS	0.0115	0.0114	0.0098	0.0102
	CvM	0.0111	0.0110	0.0102	0.0111
	AD	0.0105	0.0103	0.0106	0.0105
40	KS	0.0097	0.0107	0.0099	0.0103
	CvM	0.0089	0.0103	0.0106	0.0110
	AD	0.0090	0.0107	0.0098	0.0118
50	KS	0.0087	0.0098	0.0097	0.0106
	CvM	0.0106	0.0085	0.0096	0.0095
	AD	0.0102	0.0091	0.0091	0.0090
100	KS	0.0096	0.0094	0.0111	0.0095
	CvM	0.0092	0.0100	0.0111	0.0107
	AD	0.0095	0.0104	0.0112	0.0104

5. POWER STUDY

Power of the modified KS, CvM and AD tests are calculated and compared for sample size $n = 10(10)50$ and 100 for selected alternative distributions. The selected distributions are Weibull distribution with shape parameter $\gamma = 1,3$, scale parameter $\beta = 1$ and log-normal distribution with shape parameter $\sigma = 0.5, 0.8, 1.0$ and scale parameter $\mu = 0$. The null hypothesis of Rayleigh distribution with unspecified parameter is tested at significant value 0.05 and 0.01. For each selected test, sample size and the alternative distribution, 10,000 random samples are generated from the alternative distribution and the tests are conducted using the critical values in this paper. The proportions of rejections are recorded as the power for that situation and reported in Table 6-7.

From Table 6 and Table 7, it can be seen that power of the three tests increase as sample size increase. However, when the sample size is small ($n = 10$) they perform quite poor. Overall, the modified AD test has more power than the others when the alternative distribution is more skewed (Weibull distribution with $\gamma = 1$ and log-normal distribution with $\sigma = 0.8, 1.0$). When the alternative distribution is less skewed, the modified CvM has the biggest power. In addition, for the sample size equals to 100, the modified AD is the most powerful test for all cases.

6. CONCLUSION

In this paper, the critical values tables for modified Kolmogorov-Smirnov, Cramer-von-Mises and Anderson-Darling tests for Rayleigh distribution are obtained at various significant levels and sample sizes. For the power study of these tests by using critical values in the obtained critical values tables, in general, the modified Kolmogorov-Smirnov test has the smallest power values in all cases, while the modified Anderson-Darling test has the largest power when sample size is 100. However, when the sample size is smaller than 100, skewness of the alternative distribution plays an important role. For the more skewed distribution, the modified Anderson-Darling test has more power than the others, while the modified Cramer-von-Mises has the largest power when the distribution is less skewed.

Table 6: Power of modified tests for Rayleigh distribution when the alternative distribution is Weibull distribution

		$\alpha = 0.05$		$\alpha = 0.01$	
n	Tests	$\gamma = 1$	$\gamma = 3$	$\gamma = 1$	$\gamma = 3$
10	KS	0.5693	0.2151	0.3843	0.0626
	CvM	0.6156	0.2544	0.4497	0.0729
	AD	0.7724	0.1885	0.6397	0.0296
20	KS	0.8651	0.3966	0.7220	0.1592
	CvM	0.8989	0.4807	0.7934	0.2103
	AD	0.9551	0.4367	0.9035	0.1533
30	KS	0.9642	0.5785	0.9065	0.2762
	CvM	0.9809	0.6797	0.9417	0.3903
	AD	0.9942	0.6552	0.9795	0.3381
40	KS	0.9910	0.7008	0.9666	0.4236
	CvM	0.9960	0.8083	0.9848	0.5564
	AD	0.9992	0.8011	0.9962	0.5306
50	KS	0.9988	0.8102	0.9915	0.5446
	CvM	0.9995	0.9000	0.9955	0.7033
	AD	1.0000	0.9028	0.9989	0.6885
100	KS	1.0000	0.9866	0.9999	0.9242
	CvM	1.0000	0.9980	1.0000	0.9802
	AD	1.0000	0.9980	1.0000	0.9830

Table 7: Power of modified tests for Rayleigh distribution when the alternative distribution is log-normal distribution

		$\alpha = 0.05$			$\alpha = 0.01$		
n	Tests	$\sigma = 0.5$	$\sigma = 0.8$	$\sigma = 1$	$\sigma = 0.5$	$\sigma = 0.8$	$\sigma = 1$
10	KS	0.0994	0.3836	0.6299	0.0307	0.2458	0.4684
	CvM	0.1086	0.4155	0.6669	0.0324	0.2900	0.5245
	AD	0.0856	0.4515	0.7222	0.0231	0.3175	0.5934
20	KS	0.1332	0.6549	0.8979	0.0498	0.5102	0.7995
	CvM	0.1475	0.7008	0.9205	0.0571	0.5767	0.8499
	AD	0.1334	0.7151	0.9375	0.0443	0.5935	0.8837
30	KS	0.1760	0.8275	0.9739	0.0651	0.7006	0.9407
	CvM	0.2005	0.8619	0.9841	0.0766	0.7656	0.9638
	AD	0.2003	0.8707	0.9884	0.0651	0.7764	0.9736
40	KS	0.2133	0.9087	0.9949	0.0818	0.8268	0.9837
	CvM	0.2411	0.9334	0.9973	0.0985	0.8761	0.9923
	AD	0.2635	0.9358	0.9980	0.0942	0.8797	0.9949
50	KS	0.2562	0.9614	0.9984	0.0981	0.8998	0.9947
	CvM	0.3027	0.9747	0.9993	0.1193	0.9358	0.9978
	AD	0.3434	0.9762	0.9997	0.1166	0.9370	0.9983
100	KS	0.4587	0.9994	1.0000	0.1995	0.9970	1.0000
	CvM	0.5641	0.9997	1.0000	0.2646	0.9986	1.0000
	AD	0.7054	0.9997	1.0000	0.3441	0.9988	1.0000

7. REFERENCES

[1] Afify, W. M., Ramzy, A., "Modified Goodness of Fit Tests for the Inverse Flexible Weibull Distribution", Advances and Applications in Statistics, vol. 48 no. 4, pp.257-272, 2016. DOI: 10.17654/AS048040257
[2] Aslam M., Tahir M., Hussain Z., Al-Zahrani B. "A 3-Component Mixture of Rayleigh Distributions: Properties and Estimation in Bayesian Framework", PLoS ONE 10(5): e0126183, 2015. DOI: 10.1371/journal.pone.0126183
[3] Badr, M. M., "Goodness-of-Fit Tests for the Compound Rayleigh Distribution with Application to Real Data", Heliyon, vol. 5, e0225, 2019.
[4] Jahanshahi, S. M. A., Habibirad, A., Fakoor, V., "Some New Goodness-of-fit Tests for Rayleigh Distribution", Pakistan Journal of Statistics and Operation Research, vol. 16, no. 2, pp.305-315, 2020.
[5] Johnson, N. L., Kotz, S., Balakrishnan, N., Continuous Univariate Distributions Volume 1, 2nd edition, John Wiley & Sons, New York, 1994.
[6] Lilliefors, H. W., "On the Kolmogorov-Smirnov Test for Normality with Mean and Variance Unknown", Journal of the American Statistical Association, vol. 62, no. 318, pp.399–402, 1967. DOI: 10.2307/2283970
[7] Lilliefors, H. W., "On the Kolmogorov-Smirnov Test for the Exponential Distribution with Mean Unknown", Journal of the American Statistical Association, vol. 64, no. 325, pp.387-389, 1969. DOI: 10.1080/01621459.1969.10500983
[8] Lord Rayleigh F.R.S., "XII. On the resultant of a large number of vibrations of the same pitch and of arbitrary phase", The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, vol.10 no. 60, pp.73-78, 1880. DOI: 10.1080/14786448008626893
[9] Shawky, A.I., Bahoban, R. A., "Modified Godness-of-Fit Tests for Exponentiated Gamma Distribution with Unknown Shape Parameter", InterStat, vol. 3, Jan, pp.1-17, 2009.
[10] Woodruff, B. W., Viviano, P. J., Moore, A. H., Dunne, E.J., "Modified Goodness- of-Fit Tests for Gamma Distribution with Unknown Location and Scale Parameters", IEEE Transactions on Reliability, vol. R-33, no. 3, 1984.
[11] Yen, V. C., Moore, A. H. "Modified Goodness-of-fit Test for the Laplace Distribution", Communications in Statistics - Simulation and Computation, vol. 17, no. 1, pp.275-281, 1988. DOI: 10.1080/03610918808812661