The First Isomorphism Theorem on QI-algebras

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ABSTRACT—*The aim of this paper is to construct the first isomorphism theorem of QI-homomorphism of QI-algebras. The concepts of normal QI-subalgebras and quotient QI-algebras are also investigated.*

Keywords— QI-algebra, homomorphism, isomorphism, normal, quotient

1. INTRODUCTION

In 1966, the concept of *BCK-algebras* was introduced by Y. Imai and K. Iseki [4]. Moreover, K. Iseki [5] gave the definition of *BCI-algebras* in 1980. Both of them play an important role in the study of logical algebras. Afterwards, several structures of algebras such as *BH-algebras* [6], *TM-algebras* [7] and *KU-algebras* [12] were introduced and investigated. The fundamental concepts of abstract algebra such as ideals, congruences and homomorphisms were also studied on those algebraic structures (see [1], [11], [13]). Furthermore, many generalizations of BCK-algebras were introduced by several researchers. Some examples of such algebras are *BH-algebras* [8], *B-algebras* [9] and *Q-algebras* [10]. It turns out that many properties of these kind of algebras were extensively investigated. In 2017, A. B. Saeid, H. S. Kim and A. Razaei proposed a new algebra which is a generalization of implicative BCK-algebras, called a *BI-algebra* [14]. They provided the basic properties of BI-algebras and discussed about ideals and congruence relations. The properties of ideals of BI-algebras were continuously investigated in [2]. Lately, the notion of *QI-algebras*, which is a generalization of BI-algebras, was introduced by R. K. Bandaru [3]. The concept of ideals and some basic properties were also considered. One can see more examples of research papers in this area in [15-18].

In this paper, we gave the concept of QI-homomorphisms of QI-algebras and investigated some relate properties. The relations between QI-isomorphisms and quotient QI-algebras are also provided.

2. PRELIMINARIES

In this section, we begin with the definition of a QI-algebra which is an algebra $(X,*,0)$ of type (2,0), i.e., a nonempty set X equipped with a binary operation $*$ and a constant 0. We also recall some notions and properties of QI-algebras.

Definition 2.1. [3] An algebra $(X,*,0)$ of type (2,0) is called a *QI-algebra* if

(QI1) $x*x=0$,

(QI2) $x*0=x$,

(QI3) $x*(y*(x*y))=x*y$,

for all $x, y \in X$.

The relation "$\leq$" on a QI-algebra $(X,*,0)$ is defined by $x \leq y$ if and only if $x*y=0$. From (QI1), we can immediately conclude that $\leq$ is reflexive, however $\leq$ is not a partially ordered relation.

Example 2.2. Let $X=\{0,1,2\}$ be a set with the following Cayley table.

$*$	0	1	2
0	0	2	1
1	1	0	1
2	2	2	0

Then, by using computer programming, it is easy to check that $(X,*,0)$ is a QI-algebra.

Definition 2.3. [3] A QI-algebra $(X,*,0)$ is said to be *right distributive* or *left distributive*, respectively if

$$(x*y)*z=(x*z)*(y*z) \text{ or } z*(x*y)=(z*x)*(z*y),$$

respectively, for all $x, y, z \in X$.

Example 2.4. Notice that a QI-algebra $(X,*,0)$ in Example 2.2 is not a right distributive since

$$(1*1)*1=0*1=2\neq 0=0*0=(1*1)*(1*1),$$

and $(X,*,0)$ is not left distributive QI-algebra since

$$2*(1*0)=2*1=2\neq 0=2*2=(2*1)*(2*0).$$

Example 2.5. [3] Let $Y=\{0,1,2,3\}$ be a set with the following Cayley table.

$*'$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	2
3	3	0	1	0

Then it is easy to check that $(Y,*',0)$ is a right distributive QI-algebra.

Proposition 2.6. [3] Let $(X,*,0)$ be a QI-algebra.

(i) If X is a left distributive QI-algebra, then $X=\{0\}$.

(ii) If X is a right distributive QI-algebra, then $0*x=0$ for all $x \in X$.

Definition 2.7. [3] Let $(X,*,0)$ be a QI-algebra and I be a subset of X . Then I is called an *(QI-)ideal* of X if it satisfies the following:

(I1) $0 \in X$,

(I2) for each $x, y \in X$, if $x*y \in I$ and $y \in I$ then $x \in I$.

Example 2.8. [3] Let $X=\{0,1,2\}$ be a set with the following Cayley table.

$*$	0	1	2	3
0	0	2	1	0
1	1	0	1	0
2	2	2	0	2
3	3	2	1	0

Then it is easy to check that $(X,*,0)$ is a QI-algebra. Note that $I_1=\{0,1\}$ and $I_2=\{0,1,3\}$ are ideals but $I_3=\{0,1,2\}$ is not an ideal of X .

3. MAIN RESULTS

In this section, we give the definition of normal QI-subalgebra, congruence relation and QI-homomorphism of QI-algebra. Note that such definitions were provided analogue to the definitions on BI-algebras given in [2]. The first isomorphism theorem on QI-algebras is proven at the end of this section.

Definition 3.1. Let $(X,*,0)$ be a QI-algebra. A nonempty subset S of X is called a *QI-subalgebra* of X if it is closed under the operation $*$, i.e., $x*y \in S$ for any $x, y \in S$.

Note that every QI-subalgebra contains 0 since it is nonempty and the axiom (QI1).

Definition 3.2. Let $(X,*,0)$ be a QI-algebra. A nonempty subset N of X is called a *normal subset* of X if for each $x, y, a, b \in X$ **,** $x*y, a*b \in N$ implies $(x*a)*(y*b) \in N$.

Proposition 3.3. Let N be a normal subset of a QI-algebra $(X,*,0)$. Then N is a QI-subalgebra of X **.**

Proof. Assume that N is a normal subset of X. Let $x, y \in N$. Then $x*0 = x \in N$ and $y*0 = y \in N$. Since N is normal subset of X, it follows that $x*y = (x*y)*(0*0) \in N$. Hence N is closed under $*$. Thus N is a QI-subalgebra of X **.**

□

From the above proposition, we will call a normal subset of a QI-algebra $(X,*,0)$ a *normal QI-subalgebra* X. In general, the converse of Proposition 3.3 does not hold as it was shown in the following examples.

Example 3.4. Let $X = \{0,1,2,3\}$ be a set with the following Cayley table.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	2
3	3	0	3	0

Then, by using computer programming, it is easy to check that $(X,*,0)$ is a QI-algebra. Notice that $A = \{0,1,2\}$ is a QI-subalgebra of X but it is not normal since $3*3 = 0 \in A, 2*3 = 2 \in A$ but $(3*2)*(3*3) = 3*0 = 3 \notin A$.

Example 3.5. Let $X = \{0,1,2,3\}$ be a set with the following Cayley table.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Then, by using computer programming, it is easy to check that $(X,*,0)$ is a QI-algebra and $N = \{0,1\}$ is normal. Moreover, we have that $M = \{0,1,2\}$ is a QI-subalgebra and QI-ideal of X. Since $3*3 = 0 \in M, 2*3 = 2 \in M$ and $(3*2)*(3*3) = 3*0 = 3 \notin M$ **,** we have that M is not normal.

Lemma 3.6. Let N be a normal QI-subalgebra of a QI-algebra $(X,*,0)$ and $x,y\in N$. If $x*y\in N$, then $y*x\in N$.

Proof. Assume that $x*y\in N$. Since N is QI-subalgebra of X, it follows that $y*y=0\in N$. The fact that $y*y, x*y\in N$ and N is normal implies that $y*x=(y*x)*0=(y*x)*(y*y)\in N$. □

Definition 3.7. Let N be a normal QI-subalgebra of a QI-algebra $(X,*,0)$. A relation $\sim_N$ is defined by for each $x,y\in X$,

$$x\sim_N y \text{ if and only if } x*y\in N.$$

Proposition 3.8. Let N be a normal QI-subalgebra of a QI-algebra $(X,*,0)$. Then $\sim_N$ is a congruence relation on X.

Proof. Let $x,y,z,w\in X$. Since $x*x=0\in N$, we have that $x\sim_N x$. This means that $\sim_N$ is reflexive. From Lemma 3.6, it follows that $x\sim_N y$ implies $y\sim_N x$. Thus $\sim_N$ is symmetric. To show that $\sim_N$ is transitive, assume that $x\sim_N y$ and $y\sim_N z$. Then $x*y,\, y*z\in N$. Since $\sim_N$ is symmetric, $z*y\in N$. This implies that

$$x*z=(x*z)*0=(x*z)*(y*y)\in N$$

because $x*y,\, z*y\in N$ and N is normal. Therefore, $\sim_N$ is an equivalence relation on X.

Next, we will show that $\sim_N$ is a congruence relation on X. Assume that $x\sim_N y$ and $z\sim_N w$. Then $x*y,\, y*z\in N$. Since N is normal, $(x*z)*(y*w)\in N$. That is $x*z\sim_N y*w$, as required. □

Definition 3.9. Let N be a normal QI-subalgebra of a QI-algebra $(X,*,0)$ and $x\in X$. A congruence class $[x]_N$ of X is denoted to be the set $\{y\in X : x\sim_N y\}$. Define X/N to be the set of all congruence class of X. That is

$$X/N=\{[x]_N : x\in X\}.$$

The proof of the following lemma is straightforward, we omit the proof.

Lemma 3.10. Let N be a normal QI-subalgebra of a QI-algebra $(X,*,0)$ and $x,y\in X$. Then

$$[x]_N=[y]_N \text{ if and only if } x\sim_N y$$

Theorem 3.11. Let N be a normal QI-subalgebra of a QI-algebra $(X,*,0)$. Then the binary operation $*'$ on X/N defined by

$$[x]_N *' [y]_N=[x*y]_N,$$

for all $x,y\in X$, makes $\left(X/N, *', [0]_N\right)$ into a QI-algebra. Moreover, $[0]_N=N$.

Proof. First, we will show that $*'$ is well-defined. Let $x_1,y_1,x_2,y_2\in X$ such that $[x_1]_N=[x_2]_N$ and $[y_1]_N=[y_2]_N$.

Then $x_1 \sim_N x_2$ and $y_1 \sim_N y_2$. Since $\sim_N$ is a congruence relation, $x_1 * y_1 \sim_N x_2 * y_2$. From Lemma 3.10, it can be concluded that $[x_1 * y_1]_N = [x_2 * y_2]_N$, i.e., $[x_1]_N *' [y_1]_N = [x_2]_N *' [y_2]_N$, as required.

Next, we will show that the axioms of QI-algebra are satisfied. Let $x, y \in X$.

(QI1) $[x]_N *' [x]_N = [x * x]_N = [0]_N$,

(QI2) $[x]_N *' [0]_N = [x * 0]_N = [x]_N$,

(QI3) $[x]_N *' \left([y]_N *' \left([x]_N *' [y]_N\right)\right) = \left[x * \left(y * (x * y)\right)\right]_N = [x * y]_N = [x]_N *' [y]_N$.

Moreover, $[0]_N = \{x \in X : x \sim_N 0\} = \{x \in X : x * 0 \in N\} = \{x \in X : x \in N\} = N$. □

The QI-algebra X/N discussed in the above theorem is called the *quotient QI-algebra* of X by N. Note that the normality of N is required in order to show that $\sim_N$ is a congruence relation which implies that X/N is a QI-algebra.

In order to state the isomorphism theorem, the definition of homomorphism in QI-algebra was provided as follows.

Definition 3.12. Let $(X, *, 0_X)$ and $(Y, \square, 0_Y)$ be QI-algebras. A *QI-homomorphism* is a mapping $f : X \to Y$ satisfying

$$f(x * y) = f(x) \square f(y),$$

for all $x, y \in X$. An injective QI-homomorphism is called *QI-monomorphism*, a surjective QI-homomorphism is called *QI-epimorphism*. A *QI-isomorphism* is a QI-homomorphism which is bijective. We write $X \cong Y$ if there exists a QI-isomorphism $f : X \to Y$.

The kernel of the QI-homomorphism f, denoted by $\ker f$, is the set of elements of X that map to 0_Y.

Proposition 3.13. Let N be a normal QI-subalgebra of a QI-algebra $(X, *, 0)$. Then the mapping $\pi : X \to X/N$ given by

$$\pi(x) = [x]_N,$$

for all $x \in X$, is a QI-epimorphism and $\ker \pi = N$.

Proof. Let $x, y \in X$. Then

$$\pi(x * y) = [x * y]_N = [x]_N *' [y]_N = \pi(x) *' \pi(y).$$

Hence π is a QI-homomorphism. Since

$$\pi(X) = \{\pi(x) : x \in X\} = \{[x]_N : x \in N\} = X/N,$$

π is a QI-epimorphism. □

The mapping π in the above proposition is called the *canonical homomorphism* of X onto X/N.

Proposition 3.14. Let $(X, *, 0_X)$, $(Y, \square, 0_Y)$ be QI-algebras and $f : X \to Y$ be a QI-homomorphism and $A \subseteq X$. Then

(i) $f(0_X) = 0_Y$.

(ii) If f is a QI-monomorphism, then $\ker f = \{0_X\}$.

(iii) $\ker f$ is a QI-subalgebra of X.

(iv) If A is a QI-subalgebra of X, then $f(A)$ is a QI-subalgebra of Y.

Proof. (i) $f(0_X) = f(0_X * 0_X) = f(0_X) \square f(0_X) = 0_Y$.

(ii) Assume that f is a QI-monomorphism. It follows from (i) that $0_X \in \ker f$. To show the converse inclusion, let $x \in \ker f$. Then $f(x) = 0_Y = f(0_X)$. Since f is injective, $x = 0_X$. Hence $\ker f = \{0_X\}$.

(iii) Let $x, y \in \ker f$. Then $f(x) = 0_Y = f(y)$. Thus $f(x * y) = f(x) \square f(y) = 0_Y \square 0_Y = 0_Y$. Hence $x * y \in \ker f$.

(iv) Suppose that A is a QI-subalgebra of X. Let $x, y \in f(A)$. Then $x = f(a)$ and $y = f(b)$ for some $a, b \in A$. Since A is a QI-subalgebra, $x \square y = f(a) \square f(b) = f(a * b) \in f(A)$. Hence $f(A)$ is a QI-subalgebra of Y. □

The following example shows that $\ker f$ is not normal, in general.

Example 3.15. Consider a QI-algebra in Example 3.4. Define a mapping $f : X \to X$ by $f(x) = x$ for all $x \in X$. Then f is a QI-homomorphism and $\ker f = \{0\}$, which is a QI-subalgebra of X but not normal since $2 * 1 = 0, 3 * 1 = 0$ and $(2 * 3) * (1 * 1) = 2 * 0 = 2 \notin \ker f$.

Definition 3.16. A QI-algebra $(X, *, 0)$ is said to be a *QI₁-algebra* if for each $x, y \in X$,

$$x * y = 0 = y * x \text{ implies } x = y.$$

Example 3.17. Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	2
2	2	2	0	2
3	3	3	3	0

Then, by using computer programming, it is easy to check that $(X, *, 0)$ is a QI_1-algebra.

Proposition 3.18. Let $(X, *, 0_X)$ be a QI_1-algebra, $(Y, \square, 0_Y)$ a QI-algebra and $\phi : X \to Y$ a QI-homomorphism. Then ϕ is QI-monomorphism if and only if $\ker f = \{0_X\}$.

Proof. The necessity part is Proposition 3.14 (ii). To prove the sufficiency part, assume that $\ker f = \{0_X\}$. Let $x, y \in X$ such that $\phi(x) = \phi(y)$. Then $\phi(x * y) = \phi(x) \square \phi(y) = \phi(x) \square \phi(x) = 0_Y$. That is $x * y \in \ker f$. Similarly, we can show that $y * x \in \ker f$. Since X is a QI$_1$-algebra, $x = y$. Hence ϕ is injective. □

Proposition 3.19. Let M and N be normal QI-subalgebras of a QI-algebra $(X, *, 0)$ such that $N \subseteq M$. Then M/N is a normal QI-subalgebra of X/N .

Proof. Let $[x_1]_N *' [x_2]_N, [y_1]_N *' [y_2]_N \in M/N$. Then $[x_1 * x_2]_N, [y_1 * y_2]_N \in M/N$. That is $x_1 * x_2, y_1 * y_2 \in M$. Since M is normal, $(x_1 * x_2)*(y_1 * y_2), (x_1 * y_1)*(x_2 * y_2) \in M$. Thus $[(x_1 * x_2)*(y_1 * y_2)]_N, [(x_1 * y_1)*(x_2 * y_2)]_N \in M/N$. Hence $([x_1]_N *' [x_2]_N) *' ([y_1]_N *' [y_2]_N), ([x_1]_N *' [y_1]_N) *' ([x_2]_N *' [y_2]_N) \in M/N$. Therefore, M/N is a normal QI-subalgebra of X/N . □

In Example 3.5, we have shown that a QI-ideal need not be normal.

Definition 3.20. Let I be a QI-ideal of a QI-algebra $(X, *, 0)$. Then X is called a *normal QI-ideal* of X if it is normal.

Example 3.21. Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table.

$*$	0	1	2	3
0	0	2	1	0
1	1	0	1	1
2	2	2	0	2
3	3	2	3	0

Then, by using computer programming, it is easy to check that $(X, *, 0)$ is a QI-algebra and $I = \{0, 3\}$ is a normal QI-ideal.

Proposition 3.22. Let $(X, *, 0)$ be a QI-algebra and $I \subseteq X$. Then I is a normal QI-subalgebra of X if and only if I is a normal QI-ideal of X .

Proof. The sufficiency part follows from Proposition 3.3. To prove the necessity part, let $x, y \in X$ such that $x * y \in I$ and $y \in I$. Since I is a QI-subalgebra, $0 \in I$. Since $0, y \in I$ and I is a QI-subalgebra, we have that $0 * y \in I$. Since I is normal, $x = x * 0 = (x * 0) * 0 = (x * 0) * (y * y) \in I$. Therefore, I is a QI-ideal of X . □

Proposition 3.23. Let $(X, *, 0_X)$, $(Y, \square, 0_Y)$ be QI-algebras and $f : X \to Y$ be a QI-homomorphism. Then $\ker f$ is a QI-ideal of X .

Proof. Since $f(0_X) = 0_Y$, we have that $0_X \in \ker f$. Let $x, y \in X$ such that $x * y \in \ker f$ and $y \in \ker f$. Then

$f(x) = f(x) \square 0_Y = f(x) \square f(y) = f(x * y) = 0_Y$. Thus $x \in \ker f$. Hence $\ker f$ is a QI-ideal of X .

In Example 3.15, we have shown that a kernel of a QI-homomorphism need not be normal.

Definition 3.24. Let $(X,*,0_X)$, $(Y,\square,0_Y)$ be QI-algebras and $f: X \to Y$ be a QI-homomorphism. We say that f is a normal QI-homomorphism if $\ker f$ is a normal QI-ideal of X.

Theorem 3.25. (The first isomorphism theorem on QI-algebras) Let $(X,*,0_X)$ and $(Y,\square,0_Y)$ be QI$_1$-algebras. If $\varphi: X \to Y$ be a normal QI-homomorphism, then

$$X/\ker\varphi \cong \varphi(X)$$

Proof. Since φ is a normal QI-homomorphism, $\ker\varphi$ is normal. Then $X/\ker\varphi$ is a quotient QI-algebra of X by $\ker\varphi$. Let $K = \ker\varphi$. Define a mapping $\phi: X/K \to Y$ by

$$\phi([x]_K) = \varphi(x)$$

for all $x \in X$. We will show that ϕ is well-defined. Let $[x]_K = [y]_K \in X/K$. Then $x \sim_K y$. It follows that $x*y, y*x \in K$. Thus $\varphi(x)\square\varphi(y) = \varphi(x*y) = 0 = \varphi(y*x) = \varphi(y)\square\varphi(x)$. Since Y is QI$_1$-algebra, $\varphi(x) = \varphi(y)$. That is $\phi([x]_K) = \phi([y]_K)$. Since $\phi([x]_K *' [y]_K) = \phi([x*y]_K) = \varphi(x*y) = \varphi(x)\square\varphi(y) = \phi([x]_K)\square\phi([y]_K)$, we have that ϕ is QI-homomorphism. Next, we will prove that ϕ is injective. Clearly, $[0]_K \in \ker\phi$. Let $[x]_K \in \ker\phi$. Then $\varphi(x) = \phi([x]_K) = 0_Y$. Thus $x*0 = x \in K$. That is $x \sim_K 0_X$. It follows that $[x]_K = [0_X]_K$. Hence $\ker\phi = \{[0]_K\}$. It implies by Proposition 3.18 that ϕ is QI- monomorphism. Therefore, $X/K \cong \varphi(X)$.

□

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