New Three Step Iterative Method for Solving Nonlinear Equations Using Newton-Halley Method

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ABSTRACT— ***In this paper, we present a new three step iterative method for solving solutions of the nonlinear equations in the form of $f(x)=0$. The new method is developed based on the 2nd order of Taylor series expansion with predictor-corrector technique and compared with the well-known existing methods. The comparison criterions number of iteration, reduction of the relative error graph and computational local order of convergence (CLOC). The propose method gives better results.***

Keywords— nonlinear equation, iterative method, Taylor expansion, Newton's method, Halley's method

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1. INTRODUCTION

Many problems in pure science, applied science and engineering mostly contain nonlinear equations in the form of

$$f(x)=0, \tag{1}$$

where $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is a scalar function for an open interval I. The nonlinear equations (1) may have one solution or no solution or many solutions. For example, if $f(x)=x^2-9$ then it is easy to compute two solutions; $x=-3$ and $x=3$. The nonlinear equation, $f(x)=x^2+1$, has no solution in real numbers. However, the equations involving more complicated terms, such as trigonometric, exponential, logarithm functions are called transcendental equations, i.e. $f(x)=xe^x-1$, $f(x)=\cos(x)-x^3$, cannot be computed exact solutions. Numerical methods present in this paper are methods that can be applied to find solution of such equations by producing a sequence of numbers that converge towards the solution. They require one or more initial guesses of the solution at starting iteration and then the algorithm produces a successively more accurate approximation to the exact solution.

Various numerical methods have been developed based on the famous Newton-Raphson method [1,2,6,8,9,10,11,12,13,14,16] using the 1st order Taylor's series expansion of $f(x)$ around a given initial point $x=x_0$. An approximation solution x_{n+1} for solving (1) by the iterative scheme is

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}, \quad n=0,1,2,\ldots \tag{2}$$

This method has quadratic convergence in some neighborhood of a simple solution, α ,of f . Moreover, a well-known Halley's method [4,5,6] had cubic convergence and applied the 2nd order of Taylor series expansion for $f(x)$ which was given by

$$x_{n+1}=x_n-\frac{2f(x_n)f'(x_n)}{2f'^2(x_n)-f(x_n)f''(x_n)}, \quad n=0,1,2,\ldots \tag{3}$$

Abbasbandy [1] gave an iterative method with higher-order convergence which was

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}-\frac{f^2(x_n)f''(x_n)}{2f'^3(x_n)}-\frac{f^3(x_n)f'''(x_n)}{6f'^4(x_n)}, \quad n=0,1,2,\ldots \tag{4}$$

However, Newton's and Halley's formula are explicit and one-step iterative methods, and Abbasbandy's method requires the evaluation of the 2nd and the 3rd derivatives of the function $f(x)$.

During the last 10 years, different variations of Newton's method with higher order of convergence had been studied by considering different quadrature formula such as quadratic spline, cubic spline and integral rule, see [1,7,10,11,16]. In 2000, Weerakoon and Fernando [16] modified the classical Newton's method that used rectangular rule to compute the integral to be trapezoidal approximation. Their formula was an implicit scheme

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_{n+1}) + f'(x_n)}, \quad n = 0,1,2,... \tag{5}$$

which required x_{n+1} to compute itself. Arithmetic mean Newton's method applied arithmetic mean to compute x_{n+1} and has cubic convergence as follow:

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(y_n) + f'(x_n)}, \quad y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0,1,2,... \tag{6}$$

Bahgat and Hafiz [3] used three step predictor-corrector Newton-Halley method defined by

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$y_n = w_n - \frac{2f(w_n)f'(w_n)}{2f'^2(x_n) - f(w_n)f''(w_n)},$$

$$x_{n+1} = y_n - \frac{2f(y_n)}{f'(y_n)} - \frac{f^2(y_n)f''(y_n)}{2f'^3(y_n)}, \quad n = 0,1,2,... \tag{7}$$

2. ITERATIVE METHOD

We consider the nonlinear equation (1) where the 1st and the 2nd derivative are defined as f' and f''. Then Taylor expansion of $f(x)$ is given by,

$$f(x) = f(x_n) + \frac{(x - x_n)}{1!} f'(x_n) + \frac{(x - x_n)^2}{2!} f''(x_n) + \dots\ , \tag{8}$$

where x_n is the approximate solutions of (1). Let α be the exact solutions of (1), so (8) can be written by

$$f(\alpha) = f(x_n) + \frac{(\alpha - x_n)}{1!} f'(x_n) + \frac{(\alpha - x_n)^2}{2!} f''(x_n) + \dots . \tag{9}$$

Using (1), we get

$$0 = f(x_n) + \frac{(\alpha - x_n)}{1!} f'(x_i) + \frac{(\alpha - x_n)^2}{2!} f''(x_n) + \dots . \tag{10}$$

Consider only the first three terms (2nd order) of (10) and x_{n+1} is new approximate solutions, so we get

$$0 = f(x_n) + \frac{(x_{n+1} - x_n)}{1!} f'(x_n) + \frac{(x_{n+1} - x_n)^2}{2!} f''(x_n) . \tag{11}$$

From (11), we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{(x_{n+1} - x_n)^2 f''(x_n)}{2f'(x_n)}, \tag{12}$$

and (12) is implicit. So, we apply the predictor-corrector technique and use Newton-Halley method as a predictor and (12) as a corrector. For a given initial guess x_0, find the approximate solution x_{n+1} by the iterative schemes. From (2), (3) and (12), we get

$$w_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$y_n = w_n - \frac{2f(w_n)f'(w_n)}{2f'^2(w_n) - f(w_n)f''(w_n)},$$

$$x_{n+1} = w_n - \frac{f(w_n)}{f'(w_n)} - \frac{(y_n - w_n)^2 f''(w_n)}{2f'(w_n)},, \quad n = 0,1,2,... \tag{13}$$

and (13) is called the new three step iterative method.

3. NUMERICAL RESULTS

We present some examples to illustrate the efficiency of the new developed three-step iterative methods, see Table 1. We compare the Newton's method (NM), Abbasbandy method (AM), Halley's method (HM) and Newton-Halley method (NHM). All computer programs are performed using MATLAB R2017a and have stopping criteria of $\left|\frac{x_{n+1} - x_n}{x_n}\right| < \varepsilon$ where $\varepsilon = 1\times 10^{-12}$. The following examples are used for numerical comparing:

$f_1(x) = xe^x - 1,\quad f_2(x) = x^2 - (1-x)^5,\quad f_3(x) = \cos(x) - x^3,\quad f_4(x) = x^3 - e^{-x},$

$f_5(x) = x^3 + 4x^2 - 10,\quad f_6(x) = \cos(x) - e^{-x},\quad f_7(x) = x\ln(x) - 1.2,\quad f_8(x) = x + \sin(x) - x_3.$

The computational local order of convergence (CLOC) [14,15] can be approximated using the following formula

$$\text{CLOC} \approx \frac{\ln|x_{n-1} - x_n|}{\ln|x_{n-2} - x_{n-1}|}.$$

Table 1: Example and comparison of various iterative schemes

$f(x_n)$	**Method**	**IT**	x_n	**Rel. Err.**	**CLOC**
$f_1,\ x_0 = 1.5$	NM	7	0.56714329040978384	0.0000000000e+00	1.948815
	AM	5	0.56714329040978384	0.0000000000e+00	3.792761
	HM	5	0.56714329040978384	0.0000000000e+00	3.649304
	NHM	4	0.56714329040978384	0.0000000000e+00	6.460950
	New Method	3	0.56714329040978384	8.0397424059e-13	5.130115
$f_2,\ x_0 = 1$	NM	6	0.34595481584824206	4.6532764163e-15	2.448439
	AM	5	0.34595481584824206	1.6045780746e-16	2.726473
	HM	4	0.34595481584824206	1.6045780746e-16	2.937944
	NHM	4	0.34595481584824206	0.0000000000e+00	5.849764
	New Method	4	0.34595481584824206	0.0000000000e+00	5.718453
$f_3,\ x_0 = 2$	NM	7	0.86547403310161442	2.4373044899e-15	1.996812
	AM	5	0.86547403310161442	0.0000000000e+00	4.182008
	HM	5	0.86547403310161442	0.0000000000e+00	3.443207
	NHM	4	0.86547403310161442	0.0000000000e+00	6.197483
	New Method	3	0.86547403310161442	2.5270999185e-14	3.314783
$f_4,\ x_0 = 2$	NM	7	0.77288295914921012	3.9574742807e-13	1.893860
	AM	5	0.77288295914921012	0.0000000000e+00	4.773169
	HM	5	0.77288295914921012	0.0000000000e+00	3.688557
	NHM	4	0.77288295914921012	0.0000000000e+00	6.462679
	New Method	3	0.77288295914921012	4.2375858904e-14	3.079379
$f_5,\ x_0 = 5$	NM	8	1.36523001341409690	0.0000000000e+00	6.883373
	AM	5	1.36523001341409690	0.0000000000e+00	5.623364
	HM	5	1.36523001341409690	1.3336695960e-14	8.755135
	NHM	4	1.36523001341409690	0.0000000000e+00	9.243733
	New Method	4	1.36523001341409690	0.0000000000e+00	8.314674

Table 1: (continued)

$f(x_n)$	**Method**	**IT**	x_n	**Rel. Err.**	**CLOC**
f_6, $x_0 = 7$	NM	5	7.85359327997124800	0.0000000000e+00	3.966819
	AM	5	7.85359327997124800	0.0000000000e+00	3.221965
	HM	5	7.85359327997124800	0.0000000000e+00	3.964839
	NHM	3	7.85359327997124800	0.0000000000e+00	2.898956
	New Method	3	7.85359327997124800	0.0000000000e+00	4.236135
f_7, $x_0 = 4$	NM	6	1.88808675302834360	1.1760296743e-16	3.508870
	AM	4	1.88808675302834360	1.1760296743e-16	4.442042
	HM	4	1.88808675302834360	3.1752801207e-15	4.303257
	NHM	3	1.88808675302834360	0.0000000000e+00	5.207905
	New Method	3	1.88808675302834360	0.0000000000e+00	4.415021
f_8, $x_0 = 3$	NM	8	1.31716296100603250	0.0000000000e+00	2.121228
	AM	6	1.31716296100603250	0.0000000000e+00	3.234057
	HM	6	1.31716296100603250	2.3600910145e-15	3.111416
	NHM	4	1.31716296100603250	0.0000000000e+00	4.837526
	New Method	4	1.31716296100603250	0.0000000000e+00	4.668420

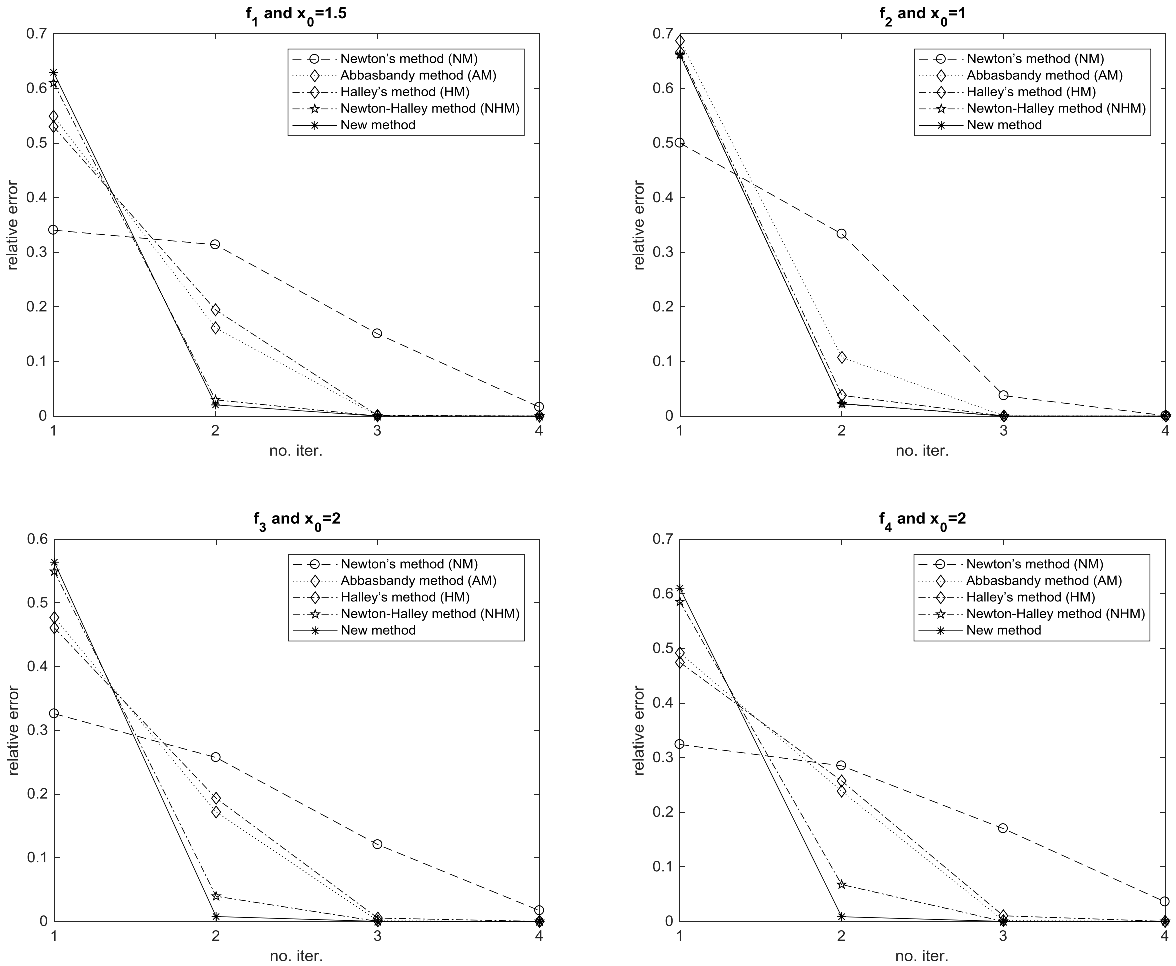

Figure 1: reduction of relative error graph of iteration number 1, 2, 3 and 4

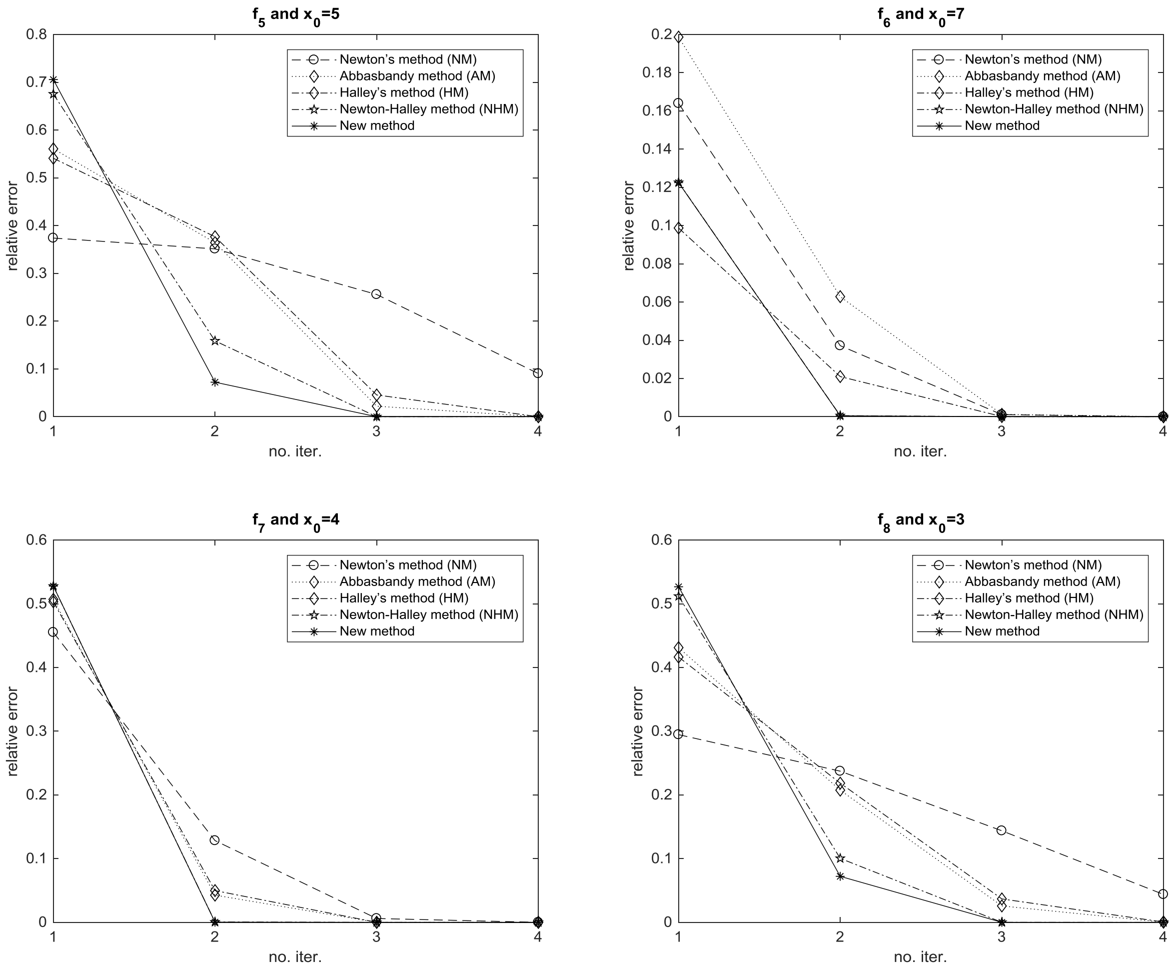

Figure 1: (continued)

4. CONCLUSIONS

In this paper, we proposed three step predictor-corrector iterative method for finding approximation solutions of nonlinear equations f_1, f_2, f_3, f_4, f_5, f_6, f_7 and f_8. The new method has the lowest number of iteration shown in Table 1. Also relative error reduces fastest among other methods as shown in Figure 1. However, computational local order of convergence (CLOC) in Table 1 can explain that the new method has a good rate of convergence for approximating solution of the functions. The approximate solution of the nonlinear equations which have multiple solutions can be considered as a new topic for the future works.

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