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Generalized Moments of Sample Extremes of Order Statistics from Discrete Uniform Distribution

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ABSTRACT---- ***More advance, it has studied in some papers that finding of first two moments of sample extremes of order statistics from discrete uniform distribution. In this paper, these moments are generalized. Also, for sample extremes of order statistics from discrete uniform distribution, moment generating functions are obtained.***

Keywords--- Binomial distribution, moment, moment generating function, orders statistics, probability density function.
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1. INTRODUCTION

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a discrete distribution and let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ denote the corresponding order statistics (Arnold et.al, 1992; David and Nagaraja, 2003).

Order statistics and their moments have assumed considerable interest in recent years. There is a vast literature on both theory and application of the moments of order statistics (Gokdere, 2014). The first two moments of order statistics from discrete distributions were proved by Khatri (1962). Arnold et al (1992) obtained the first two moments with a different way which were already obtained by Khatri (1962). All the developments on discrete order statistics lucidly accounts by Nagaraja (1992). The first two moments of sample maximum of order statistics from discrete distributions were obtained by Ahsanullah and Nevzorov (2001). For *n* up to *15,* algebraic expressions for the expected values of the sample maximum of order statistics from discrete uniform distribution were obtained by Çalik and Güngör (2004) Furthermore; *m* th raw moments of order statistics from discrete distribution were proved by Çalik et al (2010).

2. THE DISTRIBUTIONS OF ORDER STATISTICS

Suppose that $X_1, X_2, \ldots, X_n$ are random variables each with distribution function $F(x)$. Let $F_{r:n}(x),\ \ r = 1,2,\ldots,n$ denote the distribution function of the *r* th order statistics $X_{r:n}$. Then the distribution function of the largest order statistics $X_{n:n} = \max\{X_1, \ldots, X_n\}$ is given by

$$F_{n:n}(x) = P\{X_{n:n} \le x\} = P\{all\ \ X_i \le x\} = F^n(x)$$

Likewise, the distribution function of the $X_{1:n} = \min\{X_1, \ldots, X_n\}$ is

$$F_{1:n}(x) = P\{X_{1:n} \le x\} = 1 - P\{all\ \ X_i > x\} = 1 - [1 - F(x)]^n$$

The marginal distribution function $F_{r:n}(x)$ of *r* th order statistics $X_{r:n}, 1 \le r \le n$ is given

$$F_{r:n}(x) = \sum_{i=r}^{n} \binom{n}{j} [F(x)]^i [1 - F(x)]^{n-i} . \qquad (2.1)$$

From the relation between binomial series and incomplete beta function, $F_{r:n}(x)$ can be written as

$$F_{r:n}(x) = \int_0^{F(x)} \frac{n!}{(r-1)!(n-r)!} t^{r-1}(1-t)^{n-r}\, dt \tag{2.2}$$

$$F_{r:n}(x) = I_{F(x)}(r, n-r+1)$$

where $I_x(a,b) = (B(a,b))^{-1} \int_0^x t^{a-1}(1-t)^{b-1} dt$ and

$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ (David, 1981).

For discrete population, the probability mass function of $X_{r:n}$ $(1 \le r \le n)$ may be obtained from (2.2) by differencing as

$$f_{r:n}(x) = P\{X_{r:n} = x\} = F_{r:n}(x) - F_{r:n}(x-)$$

$$= C(r,n) \int_{F(x-)}^{F(x)} t^{r-1}(1-t)^{n-r}\, dt \tag{2.3}$$

where $C(r,n) = \dfrac{n!}{(r-1)!(n-r)!}$.

In particular, we also have

$f_{1:n}(x) = n \int_{F(x-)}^{F(x)} (1-t)^{n-1} dt = [\bar{F}(x-)]^n - [\bar{F}(x)]^n$, $\quad \bar{F}(x) = 1 - F(x)$ and

$f_{n:n}(x) = n \int_{F(x-)}^{F(x)} t^{n-1} dt = [F(x)]^n - [F(x-)]^n$.

3. ORDER STATISTICS FROM DISCRETE DISTRIBUTIONS

Since distributions of order statistics are more complex in discrete case, it is not easy to find their moments. Some specific approaches have been put forward for calculating these moments. Moments of order statistics of some discrete distributions can be calculated by using these approaches.

Approach- 1 (Binomial Sum). In the discrete case, which yields an expression for $F_{r:n}(x)$ for each possible value *x* of $X_{r:n}$

$$f_{r:n}(x) = F_{r:n}(x) - F_{r:n}(x-) \tag{3.1}$$

Consequently,

$$f_{r:n}(x) = \sum_{i=r}^{n} \binom{n}{i} \{[F(x)]^i [1-F(x)]^{n-i} - [F(x-)]^i [1-F(x-)]^{n-i}\} \tag{3.2}$$

Approach- 2 (Beta Integral Form). It makes use of the form of $F_{r:n}(x)$ given in (2.2) and (3.1),

$$f_{r:n}(x) = C(r;n) \int_{F(x-)}^{F(x)} u^{r-1}(1-u)^{n-r}\, du, \tag{3.3}$$

4. MOMENTS OF ORDER STATISTICS FROM DISCRETE UNIFORM DISTRIBUTION

Let $X_1, X_2, ..., X_n$ be independent and identically distributed uniform random variables from discrete population with *pmf* $f(x) = \frac{1}{k}$, *cdf* $F(x) = \frac{x}{k}$, $x = 1, ..., k.$ Then, from (3.3), *pmf* of $X_{r:n}$ can be written

$$f_{r:n}(x) = \int_{F(x-)}^{F(x)} C(r:n) u^{r-1}(1-u)^{n-r}\, du = \int_{(x-)/k}^{x/k} C(r:n) u^{r-1}(1-u)^{n-r}\, du$$

In particularly, we also have

$$f_{1:n}(x) = \left(\frac{k+1-x}{k}\right)^n - \left(\frac{k-x}{k}\right)^n, x = 1,2,...,k \tag{4.1}$$

$$f_{n:n}(x) = \left(\frac{x}{k}\right)^n - \left(\frac{x-1}{k}\right)^n, x = 1,2,...,k. \text{(Ahsanullah and Nevzorov, 2001).} \tag{4.2}$$

Theorem 1:

Let $X_1, X_2, ..., X_n$ be random variables from a discrete uniform distributions and $X_{1:n}$ be *1*th order statistics corresponding to these random variables. Then, *m* th moment of $X_{1:n}$,

whenever the moment on the left- hand side is assumed to exist,

$$\mu_{1:n}^{(m)} = E(X_{1:n}^m) = \sum_{i=1}^{k} [i^m - (i-1)^m] \left(\frac{k+1-i}{k}\right)^n.$$

Proof. If expression in (4.1) of $f_{1:n}(x)$ is written in definition of moment, *m* th moment of $X_{1:n}$,

$$\mu_{1:n}^{(m)} = E(X_{1:n}^m) = \sum_{x=1}^{k} x^m \left[\left(\frac{k+1-x}{k}\right)^n - \left(\frac{k-x}{k}\right)^n\right]$$

If the sum on the right- hand side is opened and simplification is made,

$$\mu_{1:n}^{(m)} = E(X_{1:n}^{m}) = 1^m\left[\left(\frac{k}{k}\right)^n - \left(\frac{k-1}{k}\right)^n\right] + 2^m\left[\left(\frac{k-1}{k}\right)^n - \left(\frac{k-2}{k}\right)^n\right] + \ldots + k^m\left[\left(\frac{1}{k}\right)^n\right]$$

$$= 1^m\left(\frac{k}{k}\right)^n + \left[2^m - 1^m\right]\left(\frac{k-1}{k}\right)^n + \ldots + \left[k^m - (k-1)^m\right]\left(\frac{1}{k}\right)^n$$

$$= \sum_{i=1}^{k}[i^m - (i-1)^m]\left(\frac{k+1-i}{k}\right)^n$$

Therefore, the proof is complete.

In particular, if $m=1$ and $m=2$ are taken, respectively,

$$\mu_{1:n} = E(X_{1:n}) = \sum_{i=1}^{k}\left(\frac{k+1-i}{k}\right)^n$$

$$\mu_{1:n}^{(2)} = E(X_{1:n}^{2}) = \sum_{i=1}^{k}(2i-1)\left(\frac{k+1-i}{k}\right)^n$$

(Ahsanullah and Nevzorov, 2001).

Theorem 2.

Let $X_1, X_2, \ldots, X_n$ be random variables from a discrete uniform distributions and $X_{n:n}$ be *n*th order statistics corresponding to these random variables. Then, *m* th moment of $X_{n:n}$, whenever the moment on the left- hand side is assumed to exist,

$$\mu_{n:n}^{(m)} = E(X_{n:n}^{m}) = \sum_{i=1}^{k-1}[i^m - (i+1)^m]\left(\frac{i}{k}\right)^n + k^m .$$

Proof. Omitted.

In particular, if $m=1$ and $m=2$ are taken, respectively,

$$\mu_{n:n} = E(X_{n:n}) = \sum_{i=1}^{k-1}(-1)\left(\frac{i}{k}\right)^n + k$$

$$\mu_{n:n}^{(2)} = E(X_{n:n}^{2}) = \sum_{i=1}^{k-1}(-2i-1)\left(\frac{i}{k}\right)^n + k^2 \qquad \text{(Turan, 2008).}$$

Theorem 3.

Let $X_1, X_2, \ldots, X_n$ be random variables from a discrete uniform distributions and $X_{1:n}$ be *1*th order statistics corresponding to these random variables. Then, moment generating function of $X_{1:n}$,

$$M_{X_{1:n}}(t)=\sum_{i=1}^{k}[e^{it}-e^{(i-1)t}]\left(\frac{k+1-i}{k}\right)^n \ .$$

Proof. If expression in (4.1) of $f_{1:n}(x)$ is written in definition of moment generating function, *m* th moment of $X_{1:n}$,

$$M_{X_{1:n}}(t)=\sum_{x=1}^{k}e^{tx}\left[\left(\frac{k+1-x}{k}\right)^n-\left(\frac{k-x}{k}\right)^n\right]$$

$$=e^{t}\left[\left(\frac{k}{k}\right)^n-\left(\frac{k-1}{k}\right)^n\right]+e^{2t}\left[\left(\frac{k-1}{k}\right)^n-\left(\frac{k-2}{k}\right)^n\right]+\ldots+e^{kt}\left[\left(\frac{1}{k}\right)^n\right]$$

$$=e^{t}\left[\left(\frac{k}{k}\right)^n\right]+\left[e^{2t}-e^{t}\right]\left(\frac{k-1}{k}\right)^n+\ldots+\left[e^{kt}-e^{(k-1)t}\right]\left(\frac{1}{k}\right)^n$$

$$=\sum_{i=1}^{k}[e^{it}-e^{(i-1)t}]\left(\frac{k+1-i}{k}\right)^n$$

are obtained. Therefore, the proof is complete.

Theorem 4.

Let $M_{1:n}^{(s)}(t)$ show s th derivative moment generating function of minimum order statistics from discrete uniform distribution. Then,

$$M_{1:n}^{(s)}(t)=\sum_{i=1}^{k}e^{tx}\left[i^{s}e^{it}-(i-1)^{s}e^{(i-1)t}\right]\left(\frac{k+1-i}{k}\right)^n .$$

Proof. If sequential derivatives of $M_{1:n}(t)$ are taken, in Theorem 3, the proof is completed.

Theorem 5.

Let $X_1, X_2, \ldots, X_n$ be random variables from a discrete uniform distributions and $X_{n:n}$ be *n*th order statistics corresponding to these random variables. Then, moment generating function of $X_{n:n}$,

$$M_{X_{n:n}}(t)=\sum_{i=1}^{k-1}[e^{it}-e^{(i+1)t}]\left(\frac{i}{k}\right)^n+e^{kt} .$$

Proof. Omitted.

Theorem 6.

Let $M_{n:n}^{(s)}(t)$ show s th derivative moment generating function of maximum order statistics from discrete uniform distribution. Then,

$$M_{n:n}^{(s)}(t) = \sum_{i=1}^{k-1} [i^s e^{it} - (i+1)^s e^{(i+1)t}] \left(\frac{i}{k}\right)^n + k^s e^{kt} \ .$$

Proof. Omitted.

5. RESULTS AND DISCUSSION

Because statistical theory can be easily developed for uniform distribution, make statistical inference about parameters of uniform distribution is important in terms of statistical theory.

Moments of order statistics are great importance in many statistical problems. The information obtained about the means, variances and covariances of moment of order statistics enables the evaluation of the expected values and variances of the linear functions of order statistics.

Obtained algebraic and numerical results for expected values and variances order statistics are applicated in others department. For example; in a study entitled "Natural selection and veridical perceptions", Mark et al. (2010) used the expected values of the sample maximum of order statistics from discrete uniform distributions.

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Authors' contributions
The authors read and approved the final manuscript.

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