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Laser Oscillation Model for the Coupling Phenomenon in the Co-existing of Two Population Species

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ABSTRACT — ***In the present work we report for the first time the application of the coupling phenomenon in the operation of laser to a non-physics phenomenon such as the co-existence of two species like the Hindus and Muslims in twenty three district of Assam during three decades. Relaxation oscillations and coupling phenomenon are clearly demonstrated in these cases.***

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1. INTRODUCTION

Laser oscillations as a model of sustained oscillator has been discussed in detail by Lamb Jr. and coworkers several decades ago [1-4]. One of the salient features of these discussions is their application in many non physics domains like business, coexistence of two population species, a host population and parasite population and so on. A detailed review and development of the application of the mode coupling phenomenon originally treated by Lotka [5] and Volterra [6] was made by Goel et.al. [7] . In recent years there has been growing concerns in political domain originating out of the influse of population of two species across the boarders of two countries or across the boarders two states. In view of this we consider it worthwhile to investigate the applications of the coupling phenomenon observed in the operation of loser to a non-physics phenomenon such as the co-existence of two population species

2. CLASSICAL SUSTAINED OSCILLATOR AND VANDER POLTRIODE OSCILLATOR CIRCUIT

We consider here the class of oscillators which do not decay (although they dissipate energy) and they choose their own oscillation frequencies in district contrast with the forced oscillator. Examples of these devices are triode oscillator circuits, clocks and lasers. In the section we investigate the equation of motion for an LCR circuit oscillator with a negative, saturable resistance, using the method of slowly varying amplitudes.

This technique is used in laser theory by Lamb and coworkers [1]. The result show how a sustained oscillator such as a transistor (or vacuum tube) oscillators builds up to constant amplitude determined by the condition that saturated gain equals the losses. We also discuss here a slightly different form than the usual equation which predicts a relaxation oscillation phenomenon well known in circuit theory and ruby laser. Relaxation oscillators are common place in physic but they are also seen in physiology (heart beats) and economics (business cycles). In this section we also consider the mode or frequency locking in the sustained oscillators.

Sustained Oscillators:

Suppose that a free oscillator has a net negative, saturable resistance as described by the equation of motion for its displacement v(t) (we use v for voltage inasmuch as our primary example is a triode oscillatory circuit).

$$\ddot{\upsilon} - \frac{d}{dt}\left(av - \beta^{l} v^{3}\right) + \omega^{2} v = 0 \qquad \text{………………….. (1)}$$

Where, a is the linear net gain, β^{l} is the saturation co-efficient and ω is the resonance frequency in the absence of dissipation or gain. Vander Pot [ ] obtained such an equation of motion in his treatment of triode oscillators as shown in Fig.1.

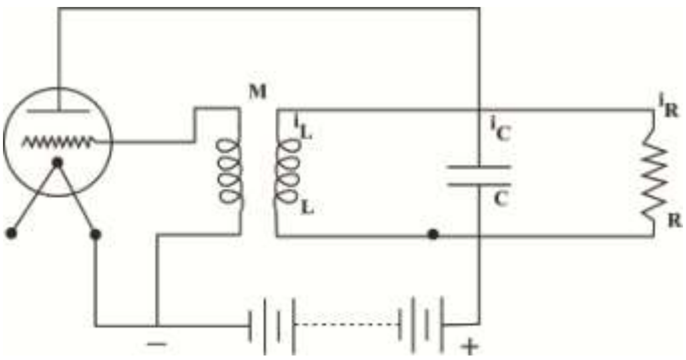

Fig 1: Triode oscillator circuit considered by Vander Pol in obtaining the equation of motion (1).

We first find a solution of the equation (1) by an import and technique known as the method of slowly varying amplitude (SVA). We suppose that the gain a is sufficiently small and the oscillator acts as undamped oscillator with frequency ω whose amplitude V(t) varies little in an optical period. Specifically we may write

$$V(t) = \frac{1}{2}(t)\exp(-i\omega t) + c.c. \qquad \text{…………….. (2)}$$

In the approximation we keep the only terms with exponential dependence $\exp(i\omega t)$. This approximation is excellent in optical frequencies for which various harmonics can exist relatively easily. In as much as both $V(t)$ and a are small quantities we neglect their product along with $\ddot{\upsilon}(t)$. In these approximations we have

$$\frac{d}{dt}\left[V(t)\exp(-i\omega t)\right] = \dot{V}\exp(-i\omega t) + V((-i\omega)\exp(-i\omega t)$$

$$\frac{d^{2}}{d+2}\left[V(t)\exp(-i\omega t)\right] = \ddot{V}\exp(-i\omega t) + \dot{V}(-i\omega)\exp(-i\omega t) + \dot{V}(-i\omega)\exp(-i\omega t) + V(-i\omega)^{2}\exp(-i\omega t)$$

$$= 2\dot{V}(-i\omega)\exp(-i\omega t) + V(-i\omega)^{2}\exp(-i\omega t)$$

$$\frac{d^{3}}{d+3}\left[V(t)\exp(-i\omega t)\right] = 2\ddot{V}(-i\omega t)\exp(-i\omega t) + 2\dot{V}(-i\omega)^{2}\exp(-i\omega t) + \dot{V}(-i\omega)^{2}\exp(-i\omega t) + V(-i\omega)^{3}\exp(-i\omega t)$$

$$= 3\dot{V}(-i\omega t)^{2}\exp(-i\omega t) + V(-i\omega)^{3}\exp(-iwt)$$

$$\left(\frac{d}{dt}\right)^{n}\left[V(t)\exp(i\omega t)\right] = \left[n(-i\omega t)\right] = \left[n(-i\omega)^{n-1}\dot{V}(-i\omega)^{n}V\right]\exp(-i\omega t) \qquad \text{………………… (3)}$$

And further, that the fundamental frequency at ω from for υ^{3} is given by

$$\upsilon^{3}\Big/_{fun} = \frac{3}{8}V^{3}\exp(-i\omega t) + c.c \qquad \text{………………… (4)}$$

Here the 3 results from the fact that are there ways to obtain $V^{3}\exp(-i\omega t)$ in the cube of Eqn (2).

Now substituting (2), (3), (4) into (1) we find

$$\ddot{\upsilon} - \frac{d}{dt}\left(a\upsilon - \beta^{/}\upsilon^{3}\right) + \omega^{3}\upsilon = 0$$

$$\ddot{\upsilon} = \frac{d^{2}}{dt^{2}}\left\{\frac{1}{2}V(t)\exp(-i\omega t)\right\}$$

$$= \frac{d}{dt}\left\{\frac{1}{2}V(t)(-i\omega)\exp(-i\omega t) + \frac{1}{2}\dot{V}(t)\exp(-i\omega t)\right\}$$

$$= \frac{1}{2}V(-i\omega)^{2}\exp(-i\omega t) + \frac{1}{2}V(t)(-i\omega)\exp(i\omega t)$$

$$+\frac{1}{2}\ddot{V}\exp(-i\omega t) + \frac{1}{2}\dot{V}(-i\omega)\exp(-i\omega t)$$

$$= \frac{1}{2}V(-i\omega)^{2}\exp(-i\omega t) - i\omega\dot{V}\exp(-i\omega t)$$

$$= \frac{1}{2}V\omega^{2}\exp(-i\omega t) - i\omega\dot{V}\exp(-i\omega t)$$

$$\frac{d}{dt}\left(a\upsilon - \beta^{/}\upsilon^{3}\right) = \frac{d}{dt}\left(a\upsilon - \frac{4}{3}\beta\upsilon^{3}\right) \quad , \quad \beta = 3\beta^{/}/4$$

$$= \frac{d}{dt}\left\{\frac{1}{2}aV\exp(-i\omega t) - \frac{4}{3}\beta\frac{3}{8}V^{3}\exp(-i\omega t)\right\}$$

$$= \frac{1}{2}a\dot{V}\exp(-i\omega t) + \frac{1}{2}aV(-i\omega)\exp(-i\omega t)$$

$$-\frac{4}{3}\beta\frac{3}{8}3V^{2}\exp(-i\omega t) - \frac{4}{3}\beta\frac{3}{8}V^{3}(-i\omega)\exp(-i\omega t)$$

$$= \frac{1}{2}aV(-i\omega)\exp(-i\omega t) + \frac{1}{2}(-i\omega)V\exp(-i\omega t) - \frac{1}{2}\beta V^{3}(-i\omega)\exp(-i\omega t)$$

And

$$\omega^{2}\upsilon = \omega^{2}\frac{1}{2}V(t)\exp(-i\omega t)$$

Thus from equation (1) we have

$$\ddot{V} - \frac{d}{dt}\left(a\upsilon - \beta^{/}\upsilon^{3}\right) + \omega^{2}\upsilon = 0$$

$$-i\omega\dot{V}\exp(-i\omega t) - \frac{1}{2}V\omega^{2}\exp(-i\omega t) + \frac{1}{2}aV(-i\omega)\exp(-i\omega t)$$

$$-\frac{1}{2}\beta V^{3}(-i\omega)\exp(-i\omega t) + \frac{1}{2}\omega^{2}V\exp(-i\omega t)$$

$$= 0$$

$$-2i\omega\dot{V} - \omega^{2}V - \left(a - \beta V^{2}\right)(-i\omega V) + \omega^{2}V = 0 \qquad \text{………………… (5)}$$

showing Eqn (5) for $\dot{V}$ we find slowly varying equation of motion

$$\dot{V}(t) = \frac{1}{2}\left(a - \beta V^2\right)V(t) \qquad \text{…………………. (6)}$$

For small $V(t)$, this equation is approximated by

$$\dot{V}(t) = \frac{1}{2}aV(t) \qquad \text{…………………. (7)}$$

With the exponentially increasing solution

$$V(t) = V(0)\exp\left(\frac{1}{2}at\right) \qquad \text{…………………. (8)}$$

For sufficiently large $V(t)$, saturation sets in, and the steady state oscillator condition

$$\dot{V}(t) = 0 \qquad \text{…………………. (9)}$$

Occurs for the value

$$V^2 = \frac{a}{\beta} \qquad \text{……………….. (10)}$$

The worthwhile point to note is that Eqn. (1) and here (6) does not build up from zero amplitude. In actual situation some sort of fluctuation initiates the thing going. In Eqn. (1) it has been assumed that the resonant frequency ω>> a , and using the decomposition (2) with V(t) a slowly varying quantity. If, on the other hand a >> ω build up processed rapidly, and in the process overshooting the steady–state value, the saturation gain becomes negative, driving the amplitude back down. This process is a repetitive one and leads to a sequence of build–up decay cycles called relaxation oscillations. These oscillations are periodic, although not sinusoidal. Relaxation oscillators occur in ruby laser operation .It may be noted that this laser has an active medium whose gain increases until it exceeds the loser due to diffraction and other factors. Laser radiation then builds up and for appropriate decay constants drives the gain down zero (atomic population in the upper and lower levels becomes equal). The losses then destroy the electric field and the process starts all over again. Relaxation oscillations exist in large numbers not only in physics, but also in physiology (heart beats) and economics as well (business cycles).

In the present work our main concern is the co-exciting of two species of population (belonging to religious groups) and the exhibition of the phenomenon of relaxation oscillation and also the coupling phenomenon in these cases, we have taken help from the data generated for this purpose.

Let us now consider two frequency operation . Van der Pol showed that the double tank (Fig 2) circuit is described by the equations.

$$\ddot{\upsilon}_1 + a_2\dot{\upsilon} - \frac{d}{dt}\left(a_1 v_1 - \beta^1 v_1^3\right) + \omega_1^2\upsilon_1 + k_1\omega_1^2\upsilon_2 = 0 \qquad \text{…………………………. (11)}$$

$$\ddot{\upsilon}_2 + a_2\dot{\upsilon}_2 + \omega_2^2\upsilon_2 + k_2\omega_2^2 v_1 = 0 \qquad \text{…………………………. (12)}$$

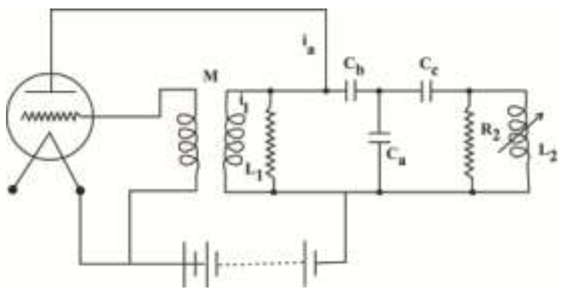

Fig. 2 : double tank circuit considered by Vander Pol in obtaining Eqns (11) and (12).

These equations describe coupled oscillators in which there is a negative saturable resistance analogous to a laser coupled to a passive Fabry Perot cavity. Such coupled systems have, sufficiently small a_1 and a_2 normal mode frequencies given by the simultaneous solutions of (11) and (12) with neglect of a_1, a_2 and β^1. By setting

$$\upsilon_1 = \upsilon_{10} \exp(-i\Omega t), \;\; \upsilon_2 = \upsilon_2 \exp(-i\Omega t) \qquad \text{......................(13)}$$

One obtain the equation

$$\Omega^4 - \left(\omega_1^2 + \omega_2^2\right)\Omega^2 + \omega_1^2\omega_2^2\left(1 - h^2\right) = 0 \qquad \text{......................(14)}$$

Where $k^2 = k_1 k_2$. Equation (14) given normal mode frequencies

$$\Omega_{1\,2}^{\;2} = \frac{1}{2}\left(\omega_1^2 + \omega_2^2\right) \pm \left[\left(\omega_1^2 - \omega_2^2\right)^2 + 4k_1^2\omega_1^2\omega_2^2\right]^{1/2} \qquad \text{......................(15)}$$

To solve (11) and (12) in the approximation of single mode treatment, we solve (11) for υ_2 in terms of υ_1 and its derivatives and use the result in (12) , there by obtaining the fourth order equation of motion .

$$\left[\left(\frac{d}{dt}\right)^2 + a_2\frac{d}{dt} + \mathrm{W}_2^2\right]\left[\left(\frac{d}{dt}\right)^2 - \frac{d}{dt}\left(a_1 - \beta^1\upsilon_1^2\right) + \omega_1^2\right]\upsilon_1 = k^2\omega_1^2\mathrm{w}_2^2\upsilon_1 \qquad \text{......................(16)}$$

We further suppose that $\upsilon_1(t)$ is a superposition of the normal modes namely

$$\upsilon_1(t) = \frac{1}{2}\mathrm{V}_1(t)\exp(-i\Omega_1 t) + \frac{1}{2}\mathrm{V}_2(t)\exp(-i\Omega_2 t) + c.c \qquad \text{...................... (17)}$$

in which V_1 and V_2 very little in oscillator periods $(2\pi \mid \Omega_1, 2\pi \mid \Omega_2)$. Therefore the derivatives of V_1 and V_2 higher first order in (16) are neglected, as well as product of first order derivatives with small constant a_1, a_2 and β^1. We further retain only the fundamental terms in υ_1^3 as in the single mode case. Thus we have

$$\upsilon_1^3\,|_{\text{fund}} = \frac{3}{8}\left[\mathrm{V}_1\exp(i\Omega_1 t) + \mathrm{V}_2\exp(i\Omega_2 t)\right] \times \left[\mathrm{V}_1\exp(-i\Omega_1 t) + \mathrm{V}_2\exp(i\Omega_2 t)^2\right]^2 + c.c \qquad \text{............ (18)}$$

in which the $\frac{1}{8}$ result from cubing the $\frac{1}{2}$ in (17) and 3 from the three possible ways of obtaining the bracketed expression in (18) from the triple product υ_1^3. Equation in (18) then becomes

$$\upsilon_1^3\,|_{\text{fund}} = \frac{3}{8}\Big\{V_1^2 + V_2^2 + V_1V_2\left(\exp\left[i(\Omega_1 - \Omega_2)t\right]\right) + \exp\left[i(\Omega_1 - \Omega_2)t\right] \times \left[V_1\exp(-i\Omega_1 t) + V_2\exp(-i\Omega_2 t)\right] + c.c$$

$$\cong \frac{3}{8}V_1\left(V_1^2 + 2V_2^2\right)\exp(-i\Omega_1 t) + \frac{3}{8}V_2\left(V_2^2 + 2V_1^2\right)\exp(-i\Omega_2 t) + c.c \qquad \text{.................. (19)}$$

It may be noted here that the "cross-coupling" term **(effect of one mode at the frequency of the other**) result from two sources. One is the dc saturation which is just like a mode inflicts upon itself. The other pulsates at the inter mode frequencies, for example exp $\left[i(\Omega_1 - \Omega_2)\right]$ which interacts with a mode term $V_1\exp(-i\Omega_2 t)$ to yield a saturation of mode two. This pulsation effect is common to many non linear problems in physics. It occurs in Raman and Brillouin scattering and in population of laser operation.

We now substitute (17) and (19) into (16) without the complex conjugates (the equations are real and only the fundamental tones are kept). Since we neglect the derivatives of the slowly varying amplitudes when multiplied by the small co-efficient a_1, a_2 and β^1, we have (in the following n, m =1, 2 with *n l m*)

$$\left(\frac{d}{dt}\right)^2 \left\{\left[a_1 V_n - \beta V_n \left(V_n^2 + 2V_m^2\right)\right]\exp\left(-i\Omega_n t\right)\right\}$$

$$= \left(-i\Omega_n\right)^k \left[a_1 V_n - \beta V_n \left(V_n^2 + 2V_m^2\right)\right]\exp\left(-i\Omega_n t\right) \qquad \text{……………….. (20)}$$

Where, $\beta = 2\beta^1 | 4$, we further note that in the slowly varying approximation of neglecting derivatives higher than first order

$$\left(\frac{d}{dt}\right)^k \left[V_n \exp\left(-i\Omega_n t\right)\right] = \left[k\left(-i\Omega_n\right)^{k-1} \dot{\upsilon}_n + \left(-i\Omega_n\right)^k V_n\right]\exp\left(\Omega_n t\right)\Big] \qquad \text{……………….. (21)}$$

With (20) and (21) the imaginary part of (16) for mode n reduces to [the real part vanishes because of (15)]

$$\left[4\Omega_n 3 - 2\Omega_n\left(\omega_1^2 + \omega_2^2\right)\right]\dot{V}_n + a_2\left(\Omega_n^3 - \Omega_n\omega_1^2\right)V_n$$

$$-\left(\Omega_n^3 - \Omega_n\omega_2^2\right)\left[a_1 - \beta\left(V_n^2 + 2V_m^2\right)\right]V_n = 0 \qquad \text{………………..(22)}$$

From (15) it may be noted that

$$\Omega_1^2 + \Omega_2^2 = \omega_1^2 + \omega_2^2 \qquad \text{……………….. (23)}$$

After multiplying by $V_n | \left[\Omega_n\left(\Omega_n - \Omega_m^2\right)\right]$ and transposing the gain and dissipation terms to the right hand side, we find (*n*, *m* = 1, 2 with $m \neq n$)

$$\frac{d}{dt}\left(V_n^2\right) = V_n^2\left(\frac{\Omega_n^2 - \omega_1^2}{\Omega_n^2 - \omega_2^2}\right)\left[a_{on} - \beta\left(V_n^2 + 2V_m^2\right)\right] \qquad \text{………………… (24)}$$

Where the net gain co-efficient

$$a_{on} = a_1 - a_2\left(\frac{\Omega_n^2 - \omega_1^2}{\Omega_n^2 - \omega_2^2}\right)$$

$$= a_1 - a_2\left[\frac{\left(\Omega_n^2 - \omega_1^2\right)^2}{k^2\omega_1^2 - \omega_2^2}\right] \qquad \text{………………… (25)}$$

Of particulars interest are the stationary solutions V_{ns} defined by values

$$\frac{d}{dt}V_n^2 = 0 \qquad \text{………………… (26)}$$

The solutions of slightly more general equations are discussed in details by Lamb and coworkers [1]. More commonly, however, both modes oscillate in laser operation , for the number 2 in Eqn (24) result from the pulsation coupling in Eqn(19). Simple saturation alone would have produced only a naturally coupled system.

3. COUPLING PHENOMENON IN POPULATION PULSATION IN ASSAM (NORTH EAST INDIA)

In the earlier section we have discussed in detail the sustained oscillator circuit following the treatment given by Van der Pol (8,9) many decades ago. We have specifically followed the treatment put forward by Sergeant et. al.(1). It is already indicated by these workers that the coupling phenomena and also the phenomena of relaxation oscillation also show up in many non physics context, such as the co-existence of two species belonging to a host population and paraside population. In the present work we are directly concerned with the population pulsations of the Hindus and Muslims in the various districts of Assam during a time intervals of decades as indicated by the official records. It is worthy of remark here the present work is the first application of the mode coupling phenomenon worked out with the help of Van der Pol's double tank circuit exhibiting sustained oscillation and coupling Fig 1 (a, b,c,d) shows the sailent features of data being plotted for 23 districts. As may be inferred from the figures there is a fairly close resemblance of the population pulsations of the Hindus and Muslims during all the four periods under consideration. It may be worthwhile to note that Goel, Mitra and Montroll have pointed out in their review (7) that competition phenomena similar to that observed in coupled oscillators do occur in the populations of biological species, political parties, business, countries , coupled reacting components in the atmosphere , bodies of water, organisms as a whole and part, components of nervous systems, elementary excitation in fluids (eddies in turbulent fluids). To some degree of approximations the population patterns in different districts of Assam resemble the phenomenon of spatial hole burnings in the gase laser theory of the semi classical theory of Laser as formulated by Lamb and coworkers [1].

4. SUMMERY AND CONCLUSION

In the present work we have discussed the coupling phenomena and also the phenomenon of relaxation oscillation exhibited by the oscillator circuit of triode valve as give by Vander Pol many decades ago. We have shown here that the coupling phenomenon also exists in a non physics situation like the co-existence of two species of population, such as the Hindus and Muslims. The following figures show it clearly.

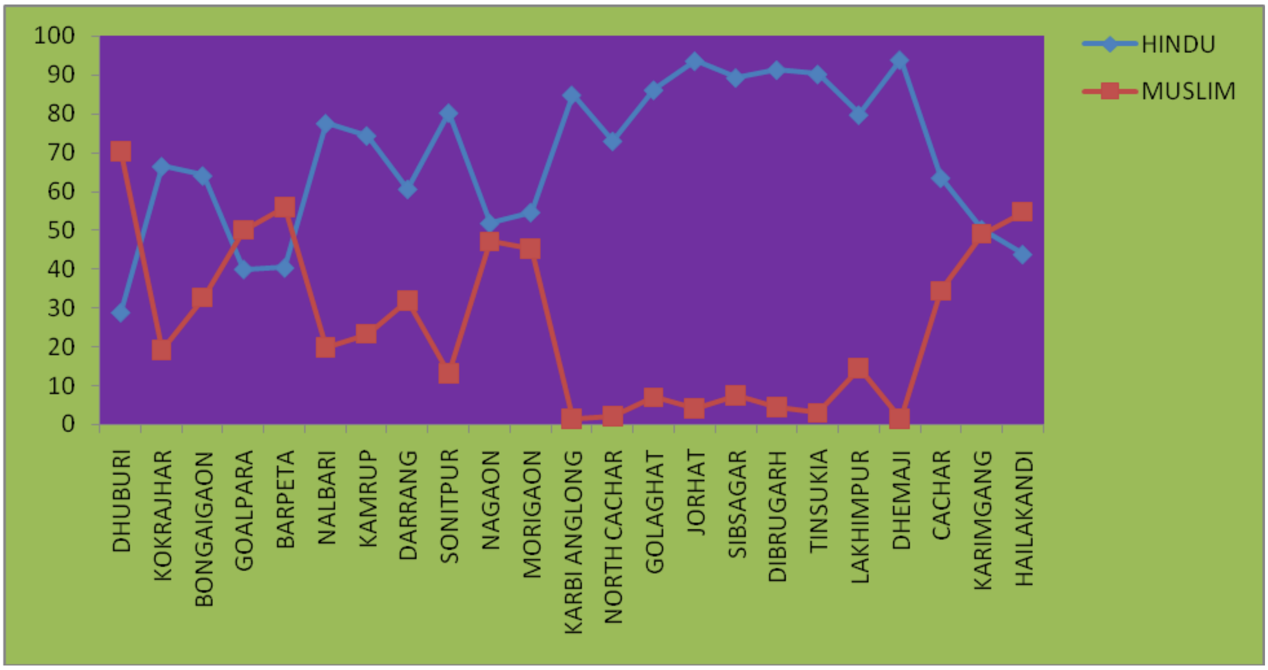

Fig 1-(a): Based on 1991 Census Data.

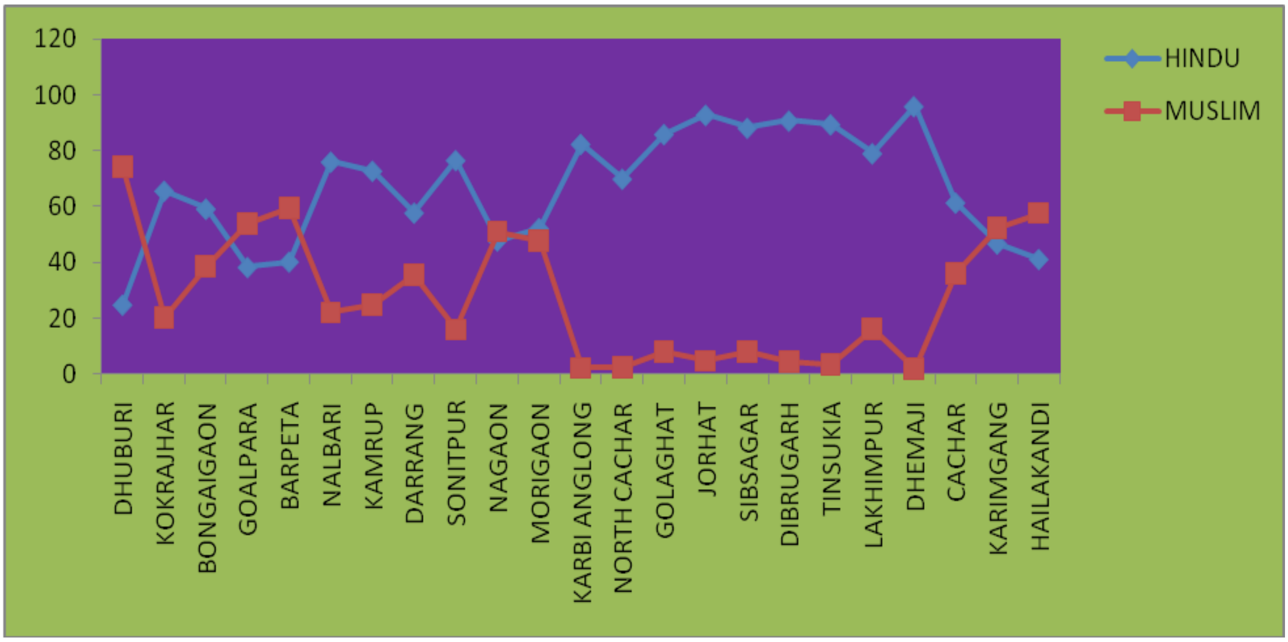

Fig 1-(b): Based on 2001 Census Data.

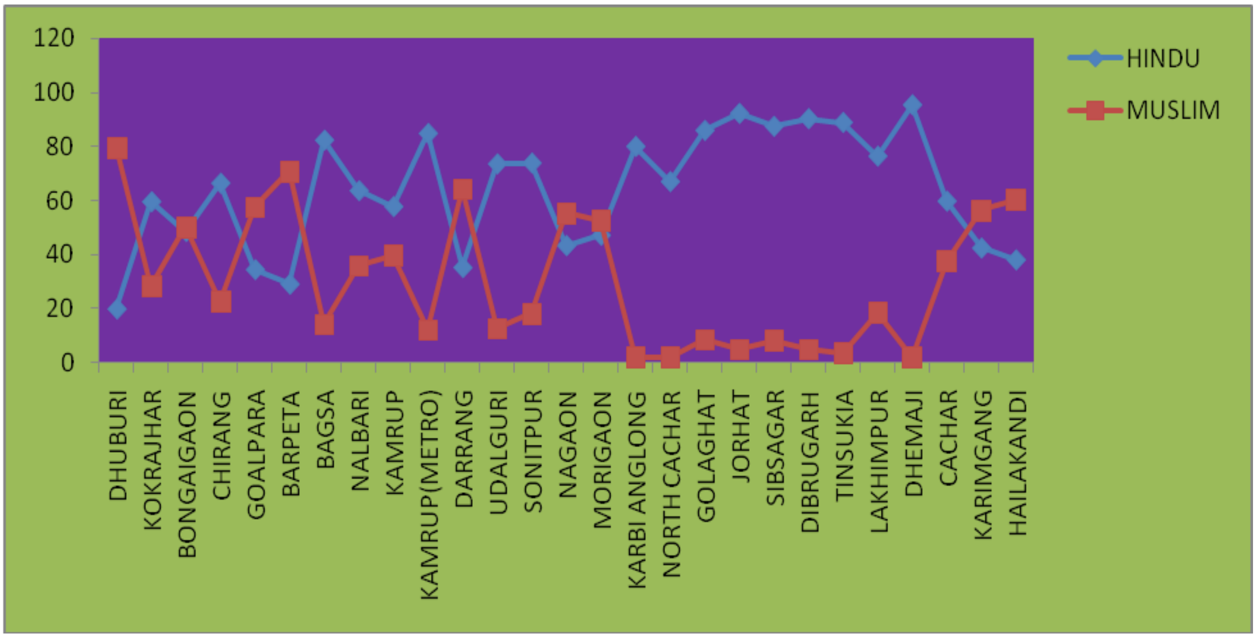

Fig 1-(c): Based on 2011 Census Data.

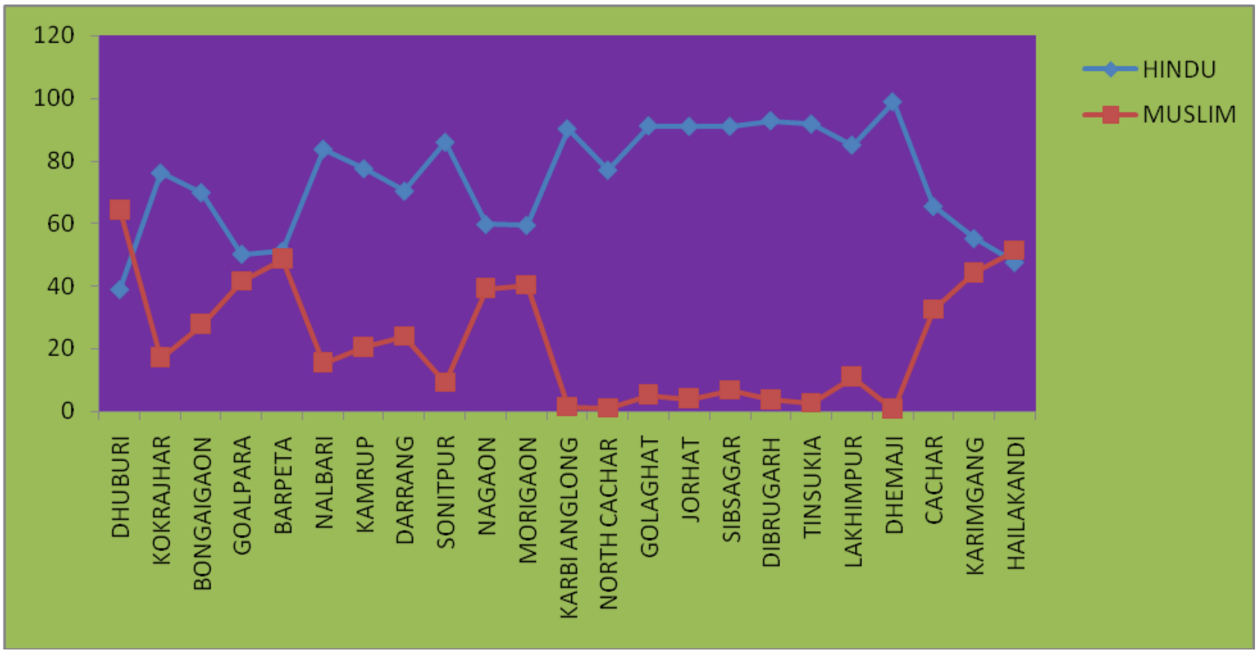

Fig 1-(d): Based on 1971 Census Data.

Fig 1(a,b,c,d) Population pulsations in various districts of Assam during several decades. The pattern resembles two mode oscillation patterns in laser theory.

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