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On the Solutions and Periods of Non-linear Recursive Systems

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ABSTRACT— ***In this paper, we obtained and examined the solutions of two rational difference equation systems. Then we studied the periodicity of these systems.***

Keywords— Non-linear difference system, periodicity of difference system

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1. INTRODUCTION

In mathematics, we can see non-linear difference equation system. Recently, many scientists have interested in many branches of mathematics as well as other sciences with difference equations system. There have been many investigations and interest in the field of functions of difference equations by several authors. Nasri et al. introduced a deterministic model for HIV infection in the presence of combination therapy related to difference equations system [1]. In [2], Cinar and Yalcinkaya studied the periodicity of positive solutions of the difference equation system

$$x_{n+1} = \frac{1}{z_n},\ y_{n+1} = \frac{1}{x_{n-1}y_{n-1}},\ z_{n+1} = \frac{1}{x_{n-1}}.$$

Grove at al. in [3], studied on the behavior and existence of the solutions of the rational equation system

$$x_{n+1} = \frac{a}{x_n} + \frac{b}{y_n},\ y_{n+1} = \frac{c}{x_n} + \frac{d}{y_n}.$$

Clark and Kulenovic, in [4], investigated the global stability properties and asymptotic behavior of solutions of the recursive system

$$x_{n+1} = \frac{x_n}{a + cy_n},\ y_{n+1} = \frac{y_n}{b + dx_n}\ \ (n = 0,1,2,\ldots).$$

Similar to references above, in [5], Kose et al investigated the solutions of following diffence equation system

$$x_{n+1} = \frac{A}{y_n},\ \ y_{n+1} = \frac{Bx_{n-1}}{x_n y_{n-1}},\ \ (n = 0,1,2,\ldots)$$

where $x_{-1}, x_0, y_{-1}, y_0, A, B \in \mathbb{R} - \{0\}$. Then they obtained equilibrium points of this system and investigated dynamics of solutions of this system.

In this study, we consider the following difference equation systems

$$x_{n+1} = \frac{y_{n-1}}{y_n(y_{n-2} + z_{n-2})} + \frac{1}{(y_{n-1} + z_{n-1})},\ y_{n+1} = \frac{1}{(y_{n-1} + z_{n-1})},\ z_{n+1} = \frac{1}{(x_{n-1} - y_{n-1})} - \frac{1}{(y_{n-1} + z_{n-1})} \tag{1.1}$$

with initial values

$x_{-1},x_0,y_{-2},y_{-1},y_0,z_{-2},z_{-1},z_0 \in \mathbb{R} -\{0\}, (y_{-2}+z_{-2}\neq 0, y_{-1}+z_{-1}\neq 0, y_0+z_0\neq 0, x_{-1}\neq y_{-1}, x_0\neq y_0)$

and

$$x_{n+1}=\frac{y_{n-1}}{y_n(y_{n-3}+z_{n-3})}+\frac{1}{(y_{n-2}+z_{n-2})},\ y_{n+1}=\frac{1}{(y_{n-2}+z_{n-2})},\ z_{n+1}=\frac{1}{(x_{n-2}-y_{n-2})}-\frac{1}{(y_{n-2}+z_{n-2})} \quad (1.2)$$

with initial values

$x_{-2},x_{-1},x_0,y_{-3},y_{-2},y_{-1},y_0,z_{-3},z_{-2},z_{-1},z_0 \in \mathbb{R} -\{0\}$, $(y_{-3}+z_{-3}\neq 0, y_{-2}+z_{-2}\neq 0, y_{-1}+z_{-1}\neq 0, y_0+z_0\neq 0,$
$x_{-2}\neq y_{-2}, x_{-1}\neq y_{-1}, x_0\neq y_0)$.

Now, we give basic, initial definitions and theorems firstly. Let I_1, I_2 and I_3 be some intervals of real numbers and let $F_1: I_2\times I_3 \to I_1$, $F_2: I_2\times I_3\to I_2$, $F_3: I_1\times I_2\times I_3\to I_3$ be three continuously differentiable functions. For every initial condition $(x_i, y_i, z_i)\in I_1\times I_2\times I_3$, it is obvious that these systems

$$x_{n+1}=F_1(y_n,z_n), \qquad y_{n+1}=F_2(y_n,z_n), \qquad z_{n+1}=F_3(x_n,y_n,z_n) \quad (1.3)$$

have a unique solution $\{x_n, y_n, z_n\}_{n=0}^{\infty}$. By using [1], we can give the following results for our study.

a) A solution $\{x_n, y_n, z_n\}_{n=0}^{\infty}$ of the system of difference equations (1.3) is periodic if there exist a positive integer *p* such that $x_{n+p}=x_n,\ y_{n+p}=y_n,\ z_{n+p}=z_n$, the smallest such positive integer *p* is called the prime period of the solution of difference equation system (1.3).

b) A point $(\bar{x},\bar{y},\bar{z})\in I_1\times I_2\times I_3$ is called an equilibrium point of system (1.3) if $\bar{x}=F_1(\bar{y},\bar{z}),\quad \bar{y}=F_2(\bar{y},\bar{z}),\quad \bar{z}=F_3(\bar{x},\bar{y},\bar{z})$.

c) The equilibrium point $(\bar{x},\bar{y},\bar{z})$ of difference equation system (1.3) is called stable (or locally stable) if for every $\varepsilon>0$, there exist $\delta>0$, such that for all $(x_s,y_s,z_s)\in I_1\times I_2\times I_3$ with $\left\|(x_s,y_s,z_s)-(\bar{x},\bar{y},\bar{z})\right\|<\delta$, implies $\left\|(x_n,y_n,z_n)-(\bar{x},\bar{y},\bar{z})\right\|<\varepsilon$ for all $n\geq 0$. Otherwise equilibrium point is called unstable.

d) The equilibrium point $(\bar{x},\bar{y},\bar{z})$ of the difference equation system (1.3) is called asymptotically stable (or locally asymptotically stable), if it is stable and there exist $\gamma>0$ such that for all $(x_s,y_s,z_s)\in I_1\times I_2\times I_3$ with $\left\|(x_s,y_s,z_s)-(\bar{x},\bar{y},\bar{z})\right\|<\gamma$, implies $\lim_{n\to\infty}\left\|(x_n,y_n,z_n)-(\bar{x},\bar{y},\bar{z})\right\|=0$.

e) The equilibrium point $(\bar{x},\bar{y},\bar{z})$ of difference equation system (1.3) is called global asymptotically stable, if it is stable and for every $(x_s,y_s,z_s)\in I_1\times I_2\times I_3$, we have $\lim_{n\to\infty}\left\|(x_n,y_n,z_n)-(\bar{x},\bar{y},\bar{z})\right\|=0$.

2. THE SOLUTIONS AND PERIODICITY OF THESE SYSTEM

Firstly we give some results related to difference equation system (1.1). In the following theorems, we show the periodicity of these solutions and obtain solutions of this system (1.1) related to initial values.

Theorem 2.1. Suppose that $\{x_n, y_n, z_n\}$ are the solutions of the difference equation system (1.1) with initial values $x_{-1}, x_0, y_{-2}, y_{-1}, y_0, z_{-2}, z_{-1}, z_0 \in \mathbb{R} - \{0\}, y_{-2} + z_{-2} \neq 0, y_{-1} + z_{-1} \neq 0, y_0 + z_0 \neq 0, x_{-1} \neq y_{-1}, x_0 \neq y_0$. Then all solutions of the system (1.1) are periodic with period 6.

Proof: From the system (1.1), it is obtained the following equalities

$$x_{n+1} = \frac{y_{n-1}}{y_n(y_{n-2}+z_{n-2})} + \frac{1}{(y_{n-1}+z_{n-1})}, \; y_{n+1} = \frac{1}{(y_{n-1}+z_{n-1})}, \; z_{n+1} = \frac{1}{(x_{n-1}-y_{n-1})} - \frac{1}{(y_{n-1}+z_{n-1})}$$

$$x_{n+2} = y_n + \frac{1}{(y_n+z_n)}, \; y_{n+2} = \frac{1}{(y_n+z_n)}, \; z_{n+2} = \frac{1}{(x_n-y_n)} - \frac{1}{(y_n+z_n)}$$

$$x_{n+3} = \frac{1}{(y_{n-1}+z_{n-1})} + x_{n-1} - y_{n-1}, \; y_{n+3} = x_{n-1} - y_{n-1}, \; z_{n+3} = \frac{y_n(y_{n-2}+z_{n-2})}{y_{n-1}} - x_{n-1} + y_{n-1}$$

$$x_{n+4} = \frac{1}{(y_n+z_n)} + x_n - y_n, \; y_{n+4} = x_n - y_n, \; z_{n+4} = \frac{1}{y_n} - x_n + y_n$$

$$x_{n+5} = x_{n-1} - y_{n-1} + \frac{y_{n-1}}{y_n(y_{n-2}+z_{n-2})}, \; y_{n+5} = \frac{y_{n-1}}{y_n(y_{n-2}+z_{n-2})}, \; z_{n+5} = y_{n-1} + z_{n-1} - \frac{y_{n-1}}{y_n(y_{n-2}+z_{n-2})}$$

$$x_{n+6} = x_n, \; y_{n+6} = y_n, \; z_{n+6} = z_n$$

Thus all solutions of the system (1.1) are periodic with 6 period.

Theorem 2.2. All solutions of the difference equation system (1.1) with initial values $x_{-1} = p, x_0 = q, y_{-2} = r, y_{-1} = s, y_0 = t, z_{-2} = u, z_{-1} = v, z_0 = w \in \mathbb{R} - \{0\}, r+u \neq 0, s+v \neq 0, t+w \neq 0, p \neq s, q \neq t$ follow

$$x_{6k+1} = \frac{s}{t(r+u)} + \frac{1}{s+v}, \; y_{6k+1} = \frac{1}{s+v}, \; z_{6k+1} = \frac{1}{p-s} - \frac{1}{s+v}$$

$$x_{6k+2} = t + \frac{1}{t+w}, \; y_{6k+2} = \frac{1}{t+w}, \; z_{6k+2} = \frac{1}{q-t} - \frac{1}{t+w}$$

$$x_{6k+3} = \frac{1}{s+v} + p - s,\ \ y_{6k+3} = p - s,\ \ z_{6k+3} = \frac{t(r+u)}{s} - p + s$$

$$x_{6k+4} = \frac{1}{t+w} + q - t,\ \ y_{6k+4} = q - t,\ \ z_{6k+4} = \frac{1}{t} - q + t$$

$$x_{6k+5} = p - s + \frac{s}{t(r+u)},\ \ y_{6k+5} = \frac{s}{t(r+u)},\ \ z_{6k+5} = s + v - \frac{s}{t(r+u)}$$

$$x_{6k+6} = x_{6k} = q,\ \ y_{6k+6} = y_{6k} = t,\ \ z_{6k+6} = z_{6k} = w.$$

Proof: By using induction method, it is obvious that above results hold for $n = 0$. Assume that these equalities hold. Now we must show that above results hold for $n = k+1$.

$$x_{6k+7} = \frac{y_{6k+5}}{y_{6k+6}(y_{6k+4} + z_{6k+4})} + \frac{1}{(y_{6k+5} + z_{6k+5})} = \frac{s}{t(r+u)} + \frac{1}{s+v},$$

$$y_{6k+7} = \frac{1}{(y_{6k+5} + z_{6k+5})} = \frac{1}{s+v},\ \ z_{6k+7} = \frac{1}{(x_{6k+5} - y_{6k+5})} - \frac{1}{(y_{6k+5} + z_{6k+5})} = \frac{1}{p-s} - \frac{1}{s+v}$$

$$x_{6k+8} = y_{6k+6} + \frac{1}{(y_{6k+6} + z_{6k+6})} = t + \frac{1}{t+w},$$

$$y_{6k+8} = \frac{1}{(y_{6k+6} + z_{6k+6})} = \frac{1}{t+w},\ \ z_{6k+8} = \frac{1}{(x_{6k+6} - y_{6k+6})} - \frac{1}{(y_{6k+6} + z_{6k+6})} = \frac{1}{q-t} - \frac{1}{t+w}$$

$$x_{6k+9} = \frac{1}{(y_{6k+5} + z_{6k+5})} + x_{6k+5} - y_{6k+5} = \frac{1}{s+v} + p - s,$$

$$y_{6k+9} = x_{6k+5} - y_{6k+5} = p - s,\ \ z_{6k+9} = \frac{y_{6k+6}(y_{6k+4} + z_{6k+4})}{y_{6k+5}} - x_{6k+5} + y_{6k+5} = \frac{t(r+u)}{s} - p + s$$

$$x_{6k+10}=\frac{1}{(y_{6k+6}+z_{6k+6})}+x_{6k+6}-y_{6k+6}=\frac{1}{t+w}+q-t,$$

$$y_{6k+10}=x_{6k+6}-y_{6k+6}=q-t,\ z_{6k+10}=\frac{1}{y_{6k+6}}-x_{6k+6}+y_{6k+6}=\frac{1}{t}-q+t$$

$$x_{6k+11}=x_{6k+5}-y_{6k+5}+\frac{y_{6k+5}}{y_{6k+6}(y_{6k+4}+z_{6k+4})}=p-s+\frac{s}{t(r+u)},$$

$$y_{6k+11}=\frac{y_{6k+5}}{y_{6k+6}(y_{6k+4}+z_{6k+4})}=\frac{s}{t(r+u)},\ z_{6k+11}=y_{6k+5}+z_{6k+5}-\frac{y_{6k+5}}{y_{6k+6}(y_{6k+4}+z_{6k+4})}=s+v-\frac{s}{t(r+u)}$$

$$x_{6k+12}=x_{6k+6}=q,\ y_{6k+12}=y_{6k+6}=t,\ z_{6k+12}=z_{6k+6}=w$$

The following theorems are related to difference equation system (1.2). Here, we obtain the periodicity of these solutions and examine solutions of the system (1.2) related to initial values.

Theorem 2.3. Suppose that $\{x_n,y_n,z_n\}$ are the solutions of the difference equation system (1.2) with initial values $x_{-2},x_{-1},x_0,y_{-3},y_{-2},y_{-1},y_0,z_{-3},z_{-2},z_{-1},z_0\in\mathbb{R}-\{0\},\ y_{-3}+z_{-3}\neq 0, y_{-2}+z_{-2}\neq 0, y_{-1}+z_{-1}\neq 0, y_0+z_0\neq 0,$ $x_{-2}\neq y_{-2}, x_{-1}\neq y_{-1}, x_0\neq y_0$. Then all solutions of the system (1.2) are periodic with period 8.

Proof: From the system (1.2), it is obtained the following equalities

$$x_{n+1}=\frac{y_{n-1}}{y_n(y_{n-3}+z_{n-3})}+\frac{1}{(y_{n-2}+z_{n-2})},\ y_{n+1}=\frac{1}{(y_{n-2}+z_{n-2})},\ z_{n+1}=\frac{1}{(x_{n-2}-y_{n-2})}-\frac{1}{(y_{n-2}+z_{n-2})}$$

$$x_{n+2}=y_n+\frac{1}{(y_{n-1}+z_{n-1})},\ y_{n+2}=\frac{1}{(y_{n-1}+z_{n-1})},\ z_{n+2}=\frac{1}{(x_{n-1}-y_{n-1})}-\frac{1}{(y_{n-1}+z_{n-1})}$$

$$x_{n+3}=\frac{1}{(y_{n-2}+z_{n-2})}+\frac{1}{(y_n+z_n)},\ y_{n+3}=\frac{1}{(y_n+z_n)},\ z_{n+3}=\frac{1}{(x_n-y_n)}-\frac{1}{(y_n+z_n)}$$

$$x_{n+4}=\frac{1}{(y_{n-1}+z_{n-1})}+x_{n-2}-y_{n-2},\ y_{n+4}=x_{n-2}-y_{n-2},\ z_{n+4}=\frac{y_n(y_{n-3}+z_{n-3})}{y_{n-1}}-(x_{n-2}-y_{n-2})$$

$$x_{n+5}=\frac{1}{(y_n+z_n)}+x_{n-1}-y_{n-1},\ y_{n+5}=x_{n-1}-y_{n-1},\ z_{n+5}=\frac{1}{y_n}-(x_{n-1}-y_{n-1})$$

$$x_{n+6} = x_{n-2} - y_{n-2} + x_n - y_n,\ y_{n+6} = x_n - y_n,\ z_{n+6} = y_{n-2} + z_{n-2} - (x_n - y_n)$$

$$x_{n+7} = x_{n-1} - y_{n-1} + \frac{y_{n-1}}{y_n(y_{n-3} + z_{n-3})},\ y_{n+7} = \frac{y_{n-1}}{y_n(y_{n-3} + z_{n-3})},\ z_{n+7} = y_{n-1} + z_{n-1} - \frac{y_{n-1}}{y_n(y_{n-3} + z_{n-3})}$$

$$x_{n+8} = x_n,\ y_{n+8} = y_n,\ z_{n+8} = z_n$$

Thus all solutions of the system (1.2) are periodic with 8 period.

Theorem 2.4. All solutions of the difference equation system (1.2) with initial values $x_{-2} = a, x_{-1} = b, x_0 = c, y_{-3} = p, y_{-2} = q, y_{-1} = r, y_0 = s, z_{-3} = t, z_{-2} = u, z_{-1} = v, z_0 = w \in \mathbb{R} - \{0\}$, $a \neq q, b \neq r, c \neq s$, $p + t \neq 0, q + u \neq 0, r + v \neq 0, s + w \neq 0,$ follow

$$x_{8k+1} = \frac{r}{s(p+t)} + \frac{1}{q+u},\ y_{8k+1} = \frac{1}{q+u},\ z_{8k+1} = \frac{1}{a-q} - \frac{1}{q+u}$$

$$x_{8k+2} = s + \frac{1}{r+v},\ y_{8k+2} = \frac{1}{r+v},\ z_{8k+2} = \frac{1}{b-r} - \frac{1}{r+v}$$

$$x_{8k+3} = \frac{1}{q+u} + \frac{1}{s+w},\ y_{8k+3} = \frac{1}{s+w},\ z_{8k+3} = \frac{1}{c-s} - \frac{1}{s+w}$$

$$x_{8k+4} = \frac{1}{r+v} + a - q,\ y_{8k+4} = a - q,\ z_{8k+4} = \frac{s(p+t)}{r} - (a-q)$$

$$x_{8k+5} = \frac{1}{s+w} + b - r,\ y_{8k+5} = b - r,\ z_{8k+5} = \frac{1}{s} - (b-r)$$

$$x_{8k+6} = a - q + c - s,\ y_{8k+6} = c - s,\ z_{8k+6} = q + u - c + s$$

$$x_{8k+7} = b - r + \frac{r}{s(p+t)},\ y_{8k+7} = \frac{r}{s(p+t)},\ z_{8k+7} = r + v - \frac{r}{s(p+t)}$$

$$x_{8k+8} = c,\ y_{8k+8} = s,\ z_{8k+8} = w.$$

Proof: By using induction method, it is obvious that above results hold for $n=0$. Assume that these equalities hold. Now we must show that above results hold for $n=k+1$.

$$x_{8k+9}=\frac{y_{8k+7}}{y_{8k+8}(y_{8k+5}+z_{8k+5})}+\frac{1}{(y_{8k+6}+z_{8k+6})}=\frac{r}{s(p+t)}+\frac{1}{q+u},$$

$$y_{8k+9}=\frac{1}{(y_{8k+6}+z_{8k+6})}=\frac{1}{q+u},\ z_{8k+9}=\frac{1}{(x_{8k+6}-y_{8k+6})}-\frac{1}{(y_{8k+6}+z_{8k+6})}=\frac{1}{a-q}-\frac{1}{q+u}$$

$$x_{8k+10}=y_{8k+8}+\frac{1}{(y_{8k+7}+z_{8k+7})}=s+\frac{1}{r+v},$$

$$y_{8k+10}=\frac{1}{(y_{8k+7}+z_{8k+7})}=\frac{1}{r+v},\ z_{8k+10}=\frac{1}{(x_{8k+7}-y_{8k+7})}-\frac{1}{(y_{8k+7}+z_{8k+7})}=\frac{1}{b-r}-\frac{1}{r+v}$$

$$x_{8k+11}=\frac{1}{(y_{8k+6}+z_{8k+6})}+\frac{1}{(y_{8k+8}+z_{8k+8})}=\frac{1}{q+u}+\frac{1}{s+w},$$

$$y_{8k+11}=\frac{1}{(y_{8k+8}+z_{8k+8})}=\frac{1}{s+w},\ z_{8k+11}=\frac{1}{(x_{8k+8}-y_{8k+8})}-\frac{1}{(y_{8k+8}+z_{8k+8})}=\frac{1}{c-s}-\frac{1}{s+w},$$

$$x_{8k+12}=\frac{1}{(y_{8k+7}+z_{8k+7})}+x_{8k+6}-y_{8k+6}=\frac{1}{r+v}+a-q,$$

$$y_{8k+12}=x_{8k+6}-y_{8k+6}=a-q,\ z_{8k+12}=\frac{y_{8k+8}(y_{8k+5}+z_{8k+5})}{y_{8k+7}}-(x_{8k+6}-y_{8k+6})=\frac{s(p+t)}{r}-(a-q),$$

$$x_{8k+13}=\frac{1}{(y_{8k+8}+z_{8k+8})}+x_{8k+7}-y_{8k+7}=\frac{1}{s+w}+b-r,$$

$$y_{8k+13}=x_{8k+7}-y_{8k+7}=b-r,\ z_{8k+13}=\frac{1}{y_{8k+8}}-(x_{8k+7}-y_{8k+7})=\frac{1}{s}-(b-r)$$

$$x_{8k+14}=x_{8k+6}-y_{8k+6}+x_{8k+8}-y_{8k+8}=a-q+c-s,$$

$$y_{8k+14} = x_{8k+8} - y_{8k+8} = c - s,\ \ z_{8k+14} = y_{8k+6} + z_{8k+6} - (x_{8k+8} - y_{8k+6}) = q + u - c + s,$$

$$x_{8k+15} = x_{8k+7} - y_{8k+7} + \frac{y_{8k+7}}{y_{8k+8}(y_{8k+5} + z_{8k+5})} = b - r + \frac{r}{s(p+t)},$$

$$y_{8k+15} = \frac{y_{8k+7}}{y_{8k+8}(y_{8k+5} + z_{8k+5})} = \frac{r}{s(p+t)},\ \ z_{8k+15} = y_{8k+7} + z_{8k+7} - \frac{y_{8k+7}}{y_{8k+8}(y_{8k+5} + z_{8k+5})} = r + v - \frac{r}{s(p+t)},$$

$$x_{8k+16} = c, \qquad y_{8k+16} = s, \qquad z_{8k+16} = w.$$

3. ILLUSTRATIVE EXAMPLES

Example 3.1. Let be $x_{-1} = 1, x_0 = 2, y_{-2} = 1, y_{-1} = 3, y_0 = 4, z_{-2} = 3, z_{-1} = 3, z_0 = 2$ in (1.1). In this case the solutions of (1.1) are periodic with 6.

n	x_n	y_n	z_n
0	2	4	2
1	0.35417	0.16667	-0.333
2	4.16659	0.16667	-0.66667
3	-5.84547	-6.01214	11.345
4	-1.833	-2	2.25001
5	-5.82462	0.18752	5.81236
6	1.99988	3.99984	1.98818

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