On the Periodicity and Behavior of Solutions of High Order Non-linear System

Uslu Kemal[1,*], Salli Gokhan[2]

[1] Selcuk University, Science Faculty, Department of Mathematics
Konya, Turkey

[2] Selcuk University, Science Faculty, Department of Mathematics
Konya, Turkey

[*]*Corresponding author's email: kuslu [AT] selcuk.edu.tr*

ABSTRACT— ***Th The aim of this work is to examinate the solutions of the high order difference system*. *Firstly we obtain equilibrium points this system. Then we give the periodicity of solutions and investigate the dynamics of this system related to equilibrium points.***

Keywords— High order difference system, the equilibrium points, the dynamics of solutions of difference system

---

1. INTRODUCTION

Difference equation system arise in many branches of mathematics as well as other sciences. It is very fascinating subject because we can drive many complex behavior based on simple formulation. It is easy to understand and has many applications. This tutorial is to introduced to a simple difference equation system, its behavior, equilibrium point and stability [1,2,3,4,5,6,]. It has been seen many investigations and interest in the field of functions of difeference equations by several authors recently. Some of them are follows. Mostafa Nasri, Mehdi Dehghan and Majid Jaberi Douraki introduced a deterministic model for HIV infectionin the presence of combination therapy related to difference equations system [2]. Clark and Kulenovic, in [1], investigated the global stability properties and asymptotic behavior of solutions of the recursive sequence

$$x_{n+1} = \frac{x_n}{a + cy_n}, y_{n+1} = \frac{y_n}{b + dx_n}, (n = 0,1,2,\ldots).$$

In [3], Uslu K., and at all studied following system and examined the global stability and asymptotic behavior of the system related to equilibrium points

$$x_{n+1} = \frac{1}{y_{n-k}}, y_{n+1} = \frac{x_{n-1}}{x_n y_{n-k-1}}, (n = 0,1,2,\ldots).$$

In this study, we consider the following high order difference equation system

$$x_{n+1} = \frac{y_{n-1}}{y_n(y_{n-k-2} + z_{n-k-2})} + \frac{1}{(y_{n-k-1} + z_{n-k-1})}, y_{n+1} = \frac{1}{(y_{n-k-1} + z_{n-k-1})}, \quad (1.1)$$

$$z_{n+1} = \frac{1}{(x_{n-k-1} - y_{n-k-1})} - \frac{1}{(y_{n-k-1} + z_{n-k-1})}$$

and investigate its solutions and periodicity, for initial values $x_{-k-1}, x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0, y_{-k-2}, y_{-k-1}, y_{-k}, \ldots, y_{-1}, y_0, z_{-k-2}, z_{-k-1}, z_{-k}, \ldots, z_{-1}, z_0 \in \mathbb{R} - \{0\}$, $y_{-k-2} + z_{-k-2} \neq 0, y_{-k-1} + z_{-k-1} \neq 0, \ldots, y_0 + z_0 \neq 0, x_{-k-1} \neq y_{-k-1}, x_{-k} \neq y_{-k}, \ldots, x_0 \neq y_0$. Then we obtain equilibrium points of the difference equation system (1.1) and investigate dynamics of this system related to equilibrium points. Firstly, we give basic definitions and theorems. Let I_1, I_2 and I_3 be some intervals of real numbers and let $F_1 : I_2 \times I_3 \to I_1$, $F_2 : I_2 \times I_3 \to I_2$, $F_3 : I_1 \times I_2 \times I_3 \to I_3$ be three continuously differentiable functions. For every initial condition $(x_i, y_i, z_i) \in I_1 \times I_2 \times I_3$, it is obvious that the system of difference equations

$$x_{n+1} = F_1(y_n, z_n), \qquad y_{n+1} = F_2(y_n, z_n), \qquad z_{n+1} = F_3(x_n, y_n, z_n) \tag{1.2}$$

has a unique solution $\{x_n, y_n, z_n\}_{n=0}^{\infty}$.

a) A solution $\{x_n, y_n, z_n\}_{n=0}^{\infty}$ of the system of difference equations (1.2) is periodic if there exist a positive integer *p* such that $x_{n+p} = x_n,\ y_{n+p} = y_n,\ z_{n+p} = z_n$, the smallest such positive integer *p* is called the prime period of the solution of difference equation system (1.2).

b) A point $(\bar{x}, \bar{y}, \bar{z}) \in I_1 \times I_2 \times I_3$ is called an equilibrium point of system (1.2) if $\bar{x} = F_1(\bar{y}, \bar{z}), \quad \bar{y} = F_2(\bar{y}, \bar{z}), \quad \bar{z} = F_3(\bar{x}, \bar{y}, \bar{z})$.

c) The equilibrium point $(\bar{x}, \bar{y}, \bar{z})$ of difference equation system (1.2) is called stable (or locally stable) if for every $\varepsilon > 0$, there exist $\delta > 0$, such that for all $(x_s, y_s, z_s) \in I_1 \times I_2 \times I_3$ with $\left\| (x_s, y_s, z_s) - (\bar{x}, \bar{y}, \bar{z}) \right\| < \delta$, implies $\left\| (x_n, y_n, z_n) - (\bar{x}, \bar{y}, \bar{z}) \right\| < \varepsilon$ for all $n \geq 0$. Otherwise equilibrium point is called unstable.

d) The equilibrium point $(\bar{x}, \bar{y}, \bar{z})$ of the difference equation system (1.2) is called asymptotically stable (or locally asymptotically stable), if it is stable and there exist $\gamma > 0$ such that for all $(x_s, y_s, z_s) \in I_1 \times I_2 \times I_3$ with $\left\| (x_s, y_s, z_s) - (\bar{x}, \bar{y}, \bar{z}) \right\| < \gamma$, implies $\lim_{n\to\infty} \left\| (x_n, y_n, z_n) - (\bar{x}, \bar{y}, \bar{z}) \right\| = 0$.

e) The equilibrium point $(\bar{x}, \bar{y}, \bar{z})$ of difference equation system (1.2) is called global asymptotically stable, if it is stable and for every $(x_s, y_s, z_s) \in I_1 \times I_2 \times I_3$, we have $\lim_{n\to\infty} \left\| (x_n, y_n, z_n) - (\bar{x}, \bar{y}, \bar{z}) \right\| = 0$.

f) Let $I_1 \times I_2 \times I_3$ be an interval of real numbers. For initial values $x_{-k-1}, x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0, y_{-k-2}, y_{-k-1} \in I_1,\ y_{-k}, \ldots, y_{-1}, y_0 \in I_2,\ z_{-k-2}, z_{-k-1}, z_{-k}, \ldots, z_{-1}, z_0 \in I_3$, if we have $\lim_{n\to\infty} \{x_n, y_n, z_n\} = (\bar{x}, \bar{y}, \bar{z})$, then the equilibrium point $(\bar{x}, \bar{y}, \bar{z})$ of the system (1.2) is global attractor [2-4].

Theorem 1.1. Let $J(\bar{x}, \bar{y}, \bar{z})$ be Jacobian matrix of system of difference equations (1.2) at the equilibrium point $(\bar{x}, \bar{y}, \bar{z})$ and $P(\lambda)$ denote the characteristics polynomial of matrix $J(\bar{x}, \bar{y}, \bar{z})$. Then the followings are true:

i) If all roots of $P(\lambda)$ lie inside the open unit disk $|\lambda|<1$, then the equilibrium point $(\bar{x},\bar{y},\bar{z})$ is asymptotically stable.

ii) If all roots of $P(\lambda)$ have absolute value greater than one, then the equilibrium point $(\bar{x},\bar{y},\bar{z})$ is repeller.

2. MAIN RESULTS

In this section results has been obtained by using results in [5,6]. The following theorem show us the periodicity of solutions of the system (1.1).

Theorem 2.1. Suppose that $\{x_n, y_n, z_n\}_{n=0}^{\infty}$ are the solutions of the system (1.1) with initial values $x_{-k-1}, x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0, y_{-k-2}, y_{-k-1}, y_{-k}, \ldots, y_{-1}, y_0, z_{-k-2}, z_{-k-1}, z_{-k}, \ldots, z_{-1}, z_0 \in \mathbb{R} - \{0\}$. Then all solutions of the system (1.1), it is obtained the following equalities

$$x_{n+1} = \frac{y_{n-1}}{y_n(y_{n-k-2}+z_{n-k-2})} + \frac{1}{(y_{n-k-1}+z_{n-k-1})}, y_{n+1} = \frac{1}{(y_{n-k-1}+z_{n-k-1})},$$

$$z_{n+1} = \frac{1}{(x_{n-k-1}-y_{n-k-1})} - \frac{1}{(y_{n-k-1}+z_{n-k-1})},$$

$$x_{n+k+1} = \frac{y_{n+k-1}}{y_{n+k}(y_{n-2}+z_{n-2})} + \frac{1}{(y_{n-1}+z_{n-1})}, \qquad y_{n+k+1} = \frac{1}{(y_{n-1}+z_{n-1})},$$

$$z_{n+k+1} = \frac{1}{(x_{n-1}-y_{n-1})} - \frac{1}{(y_{n-1}+z_{n-1})},$$

$$x_{n+k+2} = y_{n+k} + \frac{1}{(y_n+z_n)}, \qquad y_{n+k+2} = \frac{1}{(y_n+z_n)},$$

$$z_{n+k+2} = \frac{1}{(x_n-y_n)} - \frac{1}{(y_n+z_n)},$$

$$x_{n+k+3} = y_{n+k+1} + x_{n-k-1} - y_{n-k-1}, \qquad y_{n+k+3} = x_{n-k-1} - y_{n-k-1},$$

$$z_{n+k+3} = \frac{y_n(y_{n-k-2}+z_{n-k-2})}{y_{n-1}} - x_{n-k-1} + y_{n-k-1},$$

$$x_{n+k+4} = \frac{1}{y_n+z_n} + x_{n-k} - y_{n-k}, \qquad y_{n+k+4} = x_{n-k} - y_{n-k},$$

$$z_{n+k+4} = \frac{1}{y_n} - x_{n-k} + y_{n-k},$$

$$x_{n+k+5} = x_{n-k-1} - y_{n-k-1} + x_{n-k+1} - y_{n-k+1}, \quad y_{n+k+5} = x_{n-k+1} - y_{n-k+1}$$

$$z_{n+k+5} = y_{n-k-1} + z_{n-k-1} - x_{n-k+1} + y_{n-k+1},$$

$$x_{n+2k+5} = x_{n-1} - y_{n-1} + x_{n+1} - y_{n+1}, \quad y_{n+2k+5} = \frac{y_{n-1}}{y_n(y_{n-k-2} + z_{n-k-2})}$$

$$z_{n+2k+5} = y_{n-1} + z_{n-1} - \frac{y_{n-1}}{y_n(y_{n-k-2} + z_{n-k-2})},$$

$$x_{n+2k+6} = x_n - y_n + y_n = x_n, \qquad y_{n+2k+6} = y_n$$

$$z_{n+2k+5} = y_n + z_n - y_n = z_n,$$

Then all solutions of the system (1.1) are periodic with (2k+6) period. Now we give the equilibrium points of the difference equation system (1.1) in the following theorems.

Theorem 1.2. The equation system (1.1) have equilibrium points which are $(2A, A, \frac{1-A^2}{A}) \in I_1 \times I_2 \times I_3$ ($A \in \mathbb{R} - \{0\}$), where I_1, I_2, I_3 are some intervals of real numbers.

Proof: For the equilibrium points of the system (1.1), we can write the following equalities from the system (1.1)

$$\bar{x} = F_1(\bar{y}, \bar{z}) = \frac{\bar{y}}{\bar{y}(\bar{y}+\bar{z})} + \frac{1}{(\bar{y}+\bar{z})}, \quad \bar{y} = F_2(\bar{y}, \bar{z}) = \frac{1}{(\bar{y}+\bar{z})}, \quad \bar{z} = F_3(\bar{x}, \bar{y}, \bar{z}) = \frac{1}{(\bar{x}-\bar{y})} - \frac{1}{(\bar{y}+\bar{z})}.$$

From above equations and $\bar{y} = A$, we obtain the result

$$(\bar{x}, \bar{y}, \bar{z}) = (2A, A, \frac{1-A^2}{A}), \ (A \in \mathbb{R} - \{0\}).$$

Theorem 1.3. The Jacobian matrix of the system (1.1) is

$$J(2A, A, \frac{1-A^2}{A}) = \begin{pmatrix} 0 & -2A^2 & -2A^2 \\ 0 & -A^2 & -A^2 \\ \frac{-1}{A^2} & \left(\frac{1}{A^2}+A^2\right) & A^2 \end{pmatrix}$$

at the equilibrium point $(\bar{x}, \bar{y}, \bar{z}) = (2A, A, \frac{1-A^2}{A})$ and the characteristics polynomial of $J(2A, A, \frac{1-A^2}{A})$ is $-\lambda^3 + \lambda = 0.$

Proof: The Jacobian matrix at the equilibrium point $(\bar{x}, \bar{y}, \bar{z}) = (2A, A, \frac{1-A^2}{A})$ is

$$J(2A, A, \frac{1-A^2}{A}) = \begin{pmatrix} \left(\frac{\partial F_1}{\partial x}\right)_{(2A,A,\frac{1-A^2}{A})} & \left(\frac{\partial F_1}{\partial y}\right)_{(2A,A,\frac{1-A^2}{A})} & \left(\frac{\partial F_1}{\partial z}\right)_{(2A,A,\frac{1-A^2}{A})} \\ \left(\frac{\partial F_2}{\partial x}\right)_{(2A,A,\frac{1-A^2}{A})} & \left(\frac{\partial F_2}{\partial y}\right)_{(2A,A,\frac{1-A^2}{A})} & \left(\frac{\partial F_2}{\partial z}\right)_{(2A,A,\frac{1-A^2}{A})} \\ \left(\frac{\partial F_3}{\partial x}\right)_{(2A,A,\frac{1-A^2}{A})} & \left(\frac{\partial F_3}{\partial y}\right)_{(2A,A,\frac{1-A^2}{A})} & \left(\frac{\partial F_3}{\partial z}\right)_{(2A,A,\frac{1-A^2}{A})} \end{pmatrix}$$

$$J(2A, A, \frac{1-A^2}{A}) = \begin{pmatrix} 0 & \left(\frac{-2}{(\bar{y}+\bar{z})^2}\right)_{(2A,A,\frac{1-A^2}{A})} & \left(\frac{-2}{(\bar{y}+\bar{z})^2}\right)_{(2A,A,\frac{1-A^2}{A})} \\ 0 & \left(\frac{-1}{(\bar{y}+\bar{z})^2}\right)_{(2A,A,\frac{1-A^2}{A})} & \left(\frac{-1}{(\bar{y}+\bar{z})^2}\right)_{(2A,A,\frac{1-A^2}{A})} \\ \left(\frac{-1}{(\bar{x}-\bar{y})^2}\right)_{(2A,A,\frac{1-A^2}{A})} & \left(\frac{1}{(\bar{x}-\bar{y})^2}+\frac{1}{(\bar{y}+\bar{z})^2}\right)_{(2A,A,\frac{1-A^2}{A})} & \frac{1}{(\bar{y}+\bar{z})^2}{}_{(2A,A,\frac{1-A^2}{A})} \end{pmatrix}$$

$$J(2A, A, \frac{1-A^2}{A}) = \begin{pmatrix} 0 & -2A^2 & -2A^2 \\ 0 & -A^2 & -A^2 \\ \frac{-1}{A^2} & \left(\frac{1}{A^2} + A^2\right) & A^2 \end{pmatrix}$$

And the characteristics polynomial of $J(2A, A, \frac{1-A^2}{A})$ is

$$\left|J - \lambda I\right| = \begin{vmatrix} -\lambda & -2A^2 & -2A^2 \\ 0 & -A^2 - \lambda & -A^2 \\ \frac{-1}{A^2} & \frac{1}{A^2} + A^2 & A^2 - \lambda \end{vmatrix} = -\lambda^3 + \lambda = 0.$$

Because all roots of characteristics polynomial of the Jacobian matrix $J(2A, A, \frac{1-A^2}{A})$ don't lie open unit disk $\left|\lambda\right| < 1$, we can not say a result about asymptotic stability of the system (1.1) according to Theorem 1.1. After above material, we can obtain the following theorem and results.

Corollary 2.1. The solutions $\left\{x_n, y_n, z_n\right\}_{n=0}^{\infty}$ of the system (1.1) is stable because of the roots $\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = -1$ of the characteristics polynomial of $J(2A, A, \frac{1-A^2}{A})$.

Theorem 1.4. If we take the initial values $x_{-k-1}, x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$, $y_{-k-2}, y_{-k-1}, y_{-k}, \ldots, y_{-1}, y_0$, z_{-k-2}, $z_{-k-1}, z_{-k}, \ldots, z_{-1}, z_0 \in \mathbb{R} - \left\{0\right\}$ as the equilibrium point $(\bar{x}, \bar{y}, \bar{z}) = (2A, A, \frac{1-A^2}{A})$, then the following statements hold:

- **a)** The system (1.1) is global attractivity,
- **b)** The system (1.1) is asymptotically stable.

Proof: All of the solutions of the system (1.1) must converge to the equilibrium point $(2A, A, \frac{1-A^2}{A}),\ (A \in \mathbb{R} - \left\{0\right\})$. Really, for the initial values $x_{-k-1}, x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$, $y_{-k-2}, y_{-k-1}, y_{-k}, \ldots, y_{-1}, y_0$, $z_{-k-2}, z_{-k-1}, z_{-k}, \ldots, z_{-1}, z_0 \in \mathbb{R} - \left\{0\right\}$, it is obvious that $\left\{x_n, y_n, z_n\right\}_{n=0}^{\infty} = (2A, A, \frac{1-A^2}{A})$. Thus the system (1.1) is global attractivity. Because the solution of system (1.1) are stable from definition of stability and

$$\lim_{n\to\infty}\{x_n, y_n, z_n\} = (2A, A, \frac{1-A^2}{A}),$$

the solutions $\{x_n, y_n, z_n\}_{n=0}^{\infty}$ the system (1.1) are asymptotically stable.

Corollary 2.2. The system (1.1) is not repeller.

Proof: By considering Theorem 1.1., it is clearly that the solutions of the system (1.1) are not repeller.

Corollary 2.3. The system (1.1) is not asymptotically stable at the equilibrium point $(2A, A, \frac{1-A^2}{A}),\ (A \in \mathbb{R} - \{0\})$.

Proof: From Theorem 1.1., we can say that the system (1.1) is not asymptotically stable at the equilibrium point $(2A, A, \frac{1-A^2}{A})$ (i.e. $\lim_{n\to\infty}\{x_n, y_n, z_n\} \neq (2A, A, \frac{1-A^2}{A})$).

REFERENCES

[1] Cinar C., Yalcinkaya I., "On the positive solutions of difference equation system $x_{n+1} = \frac{1}{\text{[illegible]}}$, $y_{n+1} = \frac{1}{\text{[illegible]}}$, $z_{n+1} =$ [illegible], International Mathematical Journal, vol. 5, 2004.

[2] Clark D., Kulenovic M.R.S., "A coupled system of rational difference equations", Computers and Mathematics with Applications, 43, 849-867, 2002.

[3] Iricanin B., Stevic S., "Some systems of non-linear difference equations of higher order with periodic solutions", Dynamic of Continuous, Discrete and Impulse Sysetems, Series A Mathematical Analysis, vol.13, 499-507, 2006.

[4] Kılıklı G., "On the solutions of a system of the rational difference equations", The graduate school of natural and applied science od Selcuk University, Ms thesis, 2011.

[5] Nasri M., Dehghan M. and Douraki M.J., Mathias R., "Study of a system of non-linear difference equations arising in a deterministic model for HIV infection", Applied Mathematics and Computation, 171, 1306-1330, 2005.

[6] Uslu K., Taskara N., Hekimoglu O., "On the periodicity and stability conditions of a non-linear system", The First International Conference on Mathematics and Statistics, American University of Sharjah, UAE, 110 pp., 2010.