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On Prime and Left Prime Ideals in Semirings

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ABSTRACT— ***The purpose of this paper is to introduce the notion of a left prime ideals in semirings, we study prime and left prime ideals in semirings. Some characterizations of prime and left prime ideals are obtained. Moreover, we investigate relationships between prime and left prime ideals in semirings.***

Keywords— : semiring, ideal, quasi-ideal, left prime, prime.

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1. INTRODUCTION

The concept of semirings was introduced by H. S. Vandiver in 1935 and has since then been studied by many authors (see, for example, [1], [2], [12], [14], [15]). After that several authors have generalized and characterized the results in many ways. By a semiring, we mean a semigroup $(S,\cdot)$ and a commutative monoid $(S,+)$ in which 0 is the additive identity and $0s=0=s0$ for all $s\in S$ both are connected by ring-like distributivity. In this paper, all semirings are considered to be semirings with zero.

In 1958, Henriksen [13] defined a more restricted class of ideals in a semiring,which he called this special kind of ideals a *k*-ideal or subtractive. In1992, M. K. Sen [16] studied certain type of ring congruence on an additive inversive semiring with the help of full *k*-ideals, He also show that the set of full k -ideals of an additive inversive semiring forms a complete lattice, which is also modular. D. D. Anderson and E. Smith [3] have introduced and studied the concept of a weakly prime ideal of an associative ring with unity.

In this paper we characterizations of prime and left prime ideals in semirings. Moreover, we investigate relationships between prime and left prime ideals in semirings.

2. BASIC RESULTS

In this section we refer to [4, 5, 9, 10, 11] for some elementary aspects and quote few definitions and examples which are essential to step up this study. For more detail we refer to the papers in the references.

Definition 2.1. [11] A nonempty set S is said to form a semiring with respect to two binary compositions, addition $(+)$ and multiplication $(\cdot)$ defined on it, if the following conditions are satisfied.

1. $(S,+)$ is a commutative semigroup with zero,
2. $(S,\cdot)$ is a semigroup,
3. for any three elements $a,b,c\in S$ the left distributive law $a(b+c)=ab+ac$ and the right distributive law $(b+c)a=ac+ca$ both hold and
4. $s0=0=0s,$ for all $s\in S.$

Definition 2.2. [11] A nonempty subset I of a semiring S is called a left ideal of S if

1. $a, b \in I$ implies $a + b \in I$ and
2. $a \in I, s \in S$ implies $sa \in I$

Similarly we can define right ideal of a semiring. A nonempty subset I of a semiring S is an ideal if it is a left as well as a right ideal of S.

Definition 2.3. [11] An ideal I of a semiring S is called a k-ideal if $b \in S$, $a + b \in I$ and $a \in I$ implies $b \in I$.

Definition 2.4. [4, 10] A prime ideal of P of S is a proper ideal of S such that, if $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$.

Definition 2.5. [4, 10] A weakly prime ideal of P of S is a proper ideal of S such that, if $\{0\} \neq AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$.

Remark It is easy to see that every prime ideal is weakly prime.

Definition 2.6. [9] A subsemigroup A of $(S, +)$ is a quasi-ideal of S if $(SA) \cap (AS) \subseteq A$.

Definition 2.7. [5] Let **.** Let S be a semiring . A subtractive ideal (= k-ideal) K is an ideal such that if x, $x + y \in K$, then $y \in K$ (so $\{0\}$ is a k-ideal of S).

3. IDEALS IN SEMIRING

The results of the following lemmas seem play an important role to study semiring; these facts will be used so frequently that normally we shall make no reference to this lemma.

Lemma 3.1. Let S be a semiring with identity and let $a \in S$. If A is a left ideal of S, then Aa is a left ideal in S.

Proof. Let S be a semiring with left identity and let $a \in S$. Now consider

$$\begin{aligned} sa + ra &= sa + ra \\ &= (s + r)a \in Aa \end{aligned}$$

and

$$\begin{aligned} S(Aa) &= (SA)a \\ &\subseteq Aa \end{aligned}$$

for all $r, s \in A$. Therefore Aa is a left ideal in S.

Corollary 3.2. Let S be a semiring with identity and let $a \in S$. If A is a right ideal of S, then aA is a right ideal in S.

Proof. This follows from Lemma 3.1.

Lemma 3.3. Let S be a semiring with identity, and let A be a left ideal of S. Then $(A:B)$ is a left ideal in S, where $(A:B)=\{a\in S: aB\subseteq A\}$.

Proof. Suppose that S is a semiring with left identity. Let $s\in S$ and let $a,b\in(A:B)$. Then $aB\subseteq A$ and $bB\subseteq A$ so that

$$\begin{aligned}(a+b)B &= (aB)+(bB)\\ &\subseteq A+A\\ &= \mathrm{A}\end{aligned}$$

and

$$\begin{aligned}(sa)B &= s(aB)\\ &\subseteq sA\\ &\subseteq A.\end{aligned}$$

Therefore $a+b\in(A:B)$ and $S(A:B)\subseteq(A:B)$. Hence $(A:B)$ is a left ideal in S.

Corollary 3.4. Let S be a semiging with identity, and let A be a left ideal of S. Then $(A:r)$ is a left ideal in S, where $(A:r)=\{a\in S: ar\in A\}$.

Proof. This follows from Lemma 3.3.

Remark Let S be a semiring and let A be left ideals of S. It is easy to verify that $(A:C)\subseteq(A:B)$, where $B\subseteq C$.

Theorem 3.5. Let S be a semiring with identity, and let A be a left ideal of S. Then $(A:B)$ is a quasi-ideal in S.

Proof. Assume that A is a left ideal of S. By Lemma 3.3, we have $(A:B)$ is a left ideal in S. Then

$$\begin{aligned}S(A:B)\cap(A:B)S &\subseteq (A:B)\cap(A:B)\\ &\subseteq (A:B).\end{aligned}$$

Hence $(A:B)$ is a quasi-ideal in S.

Theorem 3.6. Let S be a semiring with identity, and let A be a quasi-ideal of S. Then $(A:B)$ is a quasi-ideal in S.

Proof. Assume that A is a quasi-ideal of S. Let $s\in S$ and $a\in(A:B)$. Thus $aB\subseteq A$ so that

$$s(aB)\subseteq sA \text{ and } (aB)s\subseteq As.$$

Then by hypothesis, we get

$$\begin{aligned}s(aB)\cap(aB)s &\subseteq sA\cap As\\ &\subseteq A.\end{aligned}$$

Therefore $S(A:B)\cap(A:B)S\subseteq(A:B)$ and hence $(A:B)$ is a quasi-ideal in S.

Theorem 3.7. Let S be a semiring with identity, and let A be a left k-ideal of S. Then $(A:B)$ is a left k-ideal in S.

Proof. Assume that A is a left k-ideal of S. By Lemma 3.3, we have $(A:B)$ is a left ideal in S. Let $a, a+x \in (A:B)$. So that $aB \subseteq A$ and $(a+x)B \subseteq A$ that is

$$aB \subseteq A \text{ and } aB + xB \subseteq A.$$

Then, we get $xB \subseteq A$. Hence $(A:\Gamma:B)$ is a left k-ideal in S.

4. PRIME AND LEFT PRIME IDEALS

We start with the following theorem that gives a relation between prime and left prime ideal in semirings. Our starting points is the following definition:

Definition 4.1. Let A be left ideal and left B be right ideal of S. A left ideal P is called left prime if $AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$.

Remark It is easy to see that every left prime ideal is prime.

Definition 4.2. Let A be left ideal and left B be right ideal of S. A left ideal P is called weakly left prime if $\{0\} \neq AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$.

Remark It is easy to see that every left prime ideal is weakly left prime.

Lemma 4.3. Let S be a semiring with identity and P be an ideal of S. Then P is a left prime ideal of S if $a(Sb) \subseteq P$, then $a \in P$ or $b \in P$.

Proof. Let P be a left prime ideal of a semiring S with identity. Now suppose that $a(Sb) \subseteq P$. Then by hypothesis, we get

$$\begin{aligned}(Sa)(bS) &\subseteq (Sa)S(bS)\\ &= (Sa)(Sb)S\\ &= S(a(Sb))S\\ &\subseteq (SP)S\\ &\subseteq PS\\ &\subseteq P\end{aligned}$$

that is $(Sa)(bS) \subseteq P$. Then $a = ea \in Sa \subseteq P$ or $b = be \in bS \subseteq P$.

Corollary 4.4. Let S be a semiring with identity and P be an ideal of S. Then P is a weakly left prime ideal of S if $\{0\} \neq a(Sb) \subseteq P$, then $a \in P$ or $b \in P$.

Proof. This follows from Lemma 4.3.

Theorem 4.5. Let S be a semiring with identity and let $a,b \in S$. Then a left ideal P of S is left prime if and only if $ab \in P$ implies that $a \in P$ or $b \in P$.

Proof. Let P be a left ideal of a semiring with identity. Now suppose that $ab \in P$, where $a,b \in S$. Then by hypothesis, we get

$$\begin{aligned}(Sa)(bS) &= S(ab)S \\ &\subseteq SPS \\ &\subseteq P.\end{aligned}$$

So by Definition of left prime, we have $a \in P$ or $b \in P$. Conversely, assume that if $ab \in P$ implies that $a \in P$ or $b \in P$. Let A be left ideal of S. Suppose that $AB \subseteq P$, where B is a right ideal of S such that $B \subseteq S - P$. Then there exists $b \in B$ such that $b \notin P$. Now we get $ab \in P$. So by hypothesis, $a \in P$, for all $a \in A$ implies that $A \subseteq P$. Hence P is left prime ideal in S.

Corollary 4.6. Let S be a semiring with identity and let $a,b \in S$. Then a left ideal P of S is weakly left prime if and only if $0 \neq ab \in P$ implies that $a \in P$ or $b \in P$.

Proof. This follows from Theorem 4.5.

Theorem 4.7. Let S be a semiring with left identity and let A be an ideal of S. If A is a left prime ideal of S, then $(A:B)$, is a left prime ideal in S, where $B \subseteq S - A$.

Proof. Assume that A is a left prime ideal of S. By Lemma 3.3, we have $(A:B)$ is a left ideal in S. Let $ab \in (A:B)$, where $a,b \in S$. Suppose that $b \notin (A:B)$. Since $ab \in (A:B)$, we have $(ab)S \subseteq A$. So by hypothesis,

$$\begin{aligned}(Sa)(bB) &= S(ab)B \\ &\subseteq SA \\ &\subseteq A.\end{aligned}$$

By Definition of left prime, we have $a = ea \in Sa \subseteq A$ or $bB \subseteq A$, implies that $aS \subseteq AS \subseteq A$. Hence $(A:B)$ is a left prime ideal in S.

Corollary 4.8. Let S be a semiring with left identity and let A be an ideal of S. If A is a left prime ideal of S, then $(A:s)$, is a left prime ideal in S, where $s \in S - A$.

Proof. This follows from Theorem 4.7.

Corollary 4.9. Let S be a semiring with left identity and let A be an ideal of S. If A is a weakly left prime ideal of S, then $(A:B)$, is a weakly left prime ideal in S, where $B \subseteq S - A$.

Proof. This follows from Theorem 4.7.

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