On Quasi-semiprime and Semiprime Γ-Ideals in Ordered Γ-AG-Groupoids

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ABSTRACT— ***The aim of this short note is to introduce the concepts of quasi-semiprime and semiprime Γ-ideals in ordered Γ-AG-groupoids with left identity. These concepts are related to the concepts of quasi- semiprime and semiprime Γ-ideals, play an important role in studying the structure of ordered Γ-AG-groupoids, so it seems to be interesting to study them.***

Keywords— Γ-AG-Groupoid**,** Ordered Γ-AG-Groupoid, Γ-ideal, semiprime Γ-ideal, quasi-semiprime Γ-ideal.

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1. INTRODUCTION

Abel-Grassmann's groupoid (AG-groupoid) is the generalization of semigroup theory with wide range of usages in theory of flocks (Naseeruddin, 1970). The fundamentals of this non-associative algebraic structure were first discovered by Kazim and Naseeruddin (1972). AG-groupoid is a non-associative algebraic structure mid way between a groupoid and a commutative semigroup. It is interesting to note that an AG-groupoid with right identity becomes a commutative monoid (Mushtaq and Yousuf, 1978). This structure is closely related with a commutative semigroup because if an AG-groupoid contains a right identity, then it becomes a commutative monoid [10]. A left identity in an AG-groupoid is unique [10]. It is a mid structure between a groupoid and a commutative semigroup with wide range of applications in theory of flocks [11]. Ideals in AG-groupoids have been discussed in [9] and [10]. In 1981 the notion of Γ-semigroups was introduced by M. K. Sen [13] and [18]. A groupoid is called a Γ-AG-groupoid if it satisfies the left invertive law:

$$(a\gamma b)\alpha c = (c\gamma b)\alpha a$$

for all $a,b,c \in S$ and $\gamma, \alpha \in \Gamma$ (See [8]). This structure is also known as left almost semigroup (LA-semigroup). In this paper we are going to investigate some interesting properties of newly discovered classes of namely; Γ-AG-groupoid always satisfies the Γ-medial law:

$$(a\gamma b)\alpha(c\beta d) = (a\gamma c)\alpha(b\beta d)$$

for all $a,b,c,d \in S$ and $\gamma, \alpha, \beta \in \Gamma$ (See [8]), while a Γ-AG-groupoid with left identity always satisfies Γ-paramedial law:

$$(a\gamma b)\alpha(c\beta d) = (d\gamma c)\alpha(b\beta a)$$

for all $a,b,c,d \in S$ and $\gamma, \alpha, \beta \in \Gamma$ ([8]).

The concept of an ordered AG-groupoid was first given by Faisal, Naveed Yaqoob and Kostaq Hila in [2] which is infect the generalization of an ordered semigroup. In this paper we characterize the ordered Γ-AG-groupoid. We study semiprime and quasi-semiprime Γ-ideals in ordered Γ-AG-groupoids with left identity are introduced and described.

2. BASIC RESULTS

In this section we refer to [2] for some elementary aspects and quote few definitions and examples which are essential to step up this study. For more details we refer to the papers in the references.

Definition 2.1. [2] Let S be a nonempty set, "$\cdot$" a binary operation on S and "$\leq$" a relation on S. $(S,\cdot,\leq)$ is called an ordered Γ-AG-groupoid if $(S,\cdot)$ is a Γ-AG-groupoid, $(S,\cdot)$ is a partially ordered set and for all $a,b,c \in S, a \leq b$ implies that $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$, where $\gamma \in \Gamma$.

Definition 2.2. A nonempty subset A of an ordered Γ-AG-groupoid S is called a Γ-AG-subgroupoid of S if $A\Gamma A \subseteq A$.

Let $(S,\cdot,\leq)$ be an ordered Γ-AG-groupoid. For $A \subseteq S$, let $(H] = \{x \in S : x \leq a$ for some $a \in A\}$. Following lemma is similar to the case of ordered Γ-AG-groupoids.

Lemma 2.3. Let $(S,\cdot,\leq)$ be an ordered Γ-AG-groupoid with left identity and A, B subsets of S. The following statements hold:

1. If $A \subseteq B$, then $(A] \subseteq (B]$.
2. $(A]\Gamma(B] \subseteq (A\Gamma B]$.
3. $((A]\Gamma(B]] \subseteq (A\Gamma B]$.
4. $A \subseteq (A]$.
5. $((a]] = (A]$.

Proof The proof is obvious.

Definition 2.4. A nonempty subset A of an ordered Γ-AG-groupoid S is called a left Γ-ideal of S if $(A] \subseteq A$ and $S\Gamma A \subseteq A$ and called a right Γ-ideal of S if $(A] \subseteq A$ and $A\Gamma S \subseteq A$. A nonempty subset A of S is called a Γ-ideal of S if A is both left and right Γ-ideal of S.

Lemma 2.5. Let $(S,\cdot,\leq)$ be an ordered Γ-AG-groupoid with left identity. Then every right Γ-ideal of S is a left Γ-ideal of S.

Proof The proof is obvious.

Lemma 2.6. Let $(S,\cdot,\leq)$ be an ordered Γ-AG-groupoid with left identity and $A \subseteq S$. Then $S\Gamma(S\Gamma A) = S\Gamma A$ and $S\Gamma(S\Gamma A] \subseteq (S\Gamma A]$.

Proof The proof is obvious.

Lemma 2.7. If $(S,\cdot,\leq)$ is an ordered Γ-AG-groupoid with left identity and let $a \in S$, then $(S\Gamma a]$ a left Γ-ideal of S.

Proof Assume that $(S,\cdot,\leq)$ is an ordered Γ-AG-groupoid with left identity. By Lemma 2.3, we have $(S\Gamma a] = ((S\Gamma a]]$. Then

$$\begin{aligned} S\Gamma(S\Gamma a] &= (S]\Gamma(S\Gamma a] \\ &\subseteq (S\Gamma(S\Gamma a)] \\ &= (S\Gamma(S\Gamma a)] \\ &= ((a\Gamma S)\Gamma S] \end{aligned}$$

$$= ((S\Gamma S)\Gamma a]$$
$$= (S\Gamma a].$$

Therefore $(S\Gamma a]$ is a left Γ-ideal of S.

Corollary 2.8. Let $(S,\cdot,\leq)$ be an ordered Γ-AG-groupoid with left identity and $a\in S$. Then $< a >_l = (S\Gamma a]$.
Proof The proof is obvious.

Lemma 2.9. If $(S,\cdot,\leq)$ is an ordered Γ-AG-groupoid with left identity and let $a\in S$, then $(a\Gamma S]$ is a left Γ-ideal of S.
Proof Assume that $(S,\cdot,\leq)$ is an ordered Γ-AG-groupoid with left identity. By Lemma 2.3, we have $(a\Gamma S] = ((a\Gamma S]]$. Then

$$\begin{aligned} S\Gamma(a\Gamma S] &= (S]\Gamma(a\Gamma S] \\ &\subseteq (S\Gamma(a\Gamma S)] \\ &= (a\Gamma(S\Gamma S)] \\ &= (a\Gamma S]. \end{aligned}$$

Therefore $(a\Gamma S]$ is a left Γ-ideal of S.

Proposition 2.10. If $(S,\cdot,\leq)$ is an ordered Γ-AG-groupoid with left identity and let $a\in S$, then $(a^2\Gamma S]$ is a Γ-ideal of S.
Proof Assume that $(S,\cdot,\leq)$ is an ordered Γ-AG-groupoid with left identity. By Lemma 2.9, we have $(a^2\Gamma S]$ is a left Γ-ideal of S. Then

$$\begin{aligned} (a^2\Gamma S]\Gamma S &= (a^2\Gamma S]\Gamma(S] \\ &\subseteq ((a^2\Gamma S)\Gamma S] \\ &= ((S\Gamma S)\Gamma a^2] \\ &= (a^2\Gamma S]. \end{aligned}$$

Therefore $(a^2\Gamma S]$ is a Γ-ideal of S.

Lemma 2.11. If $(S,\cdot,\leq)$ is an ordered Γ-AG-groupoid with left identity and let $a\in S$, then $(a\cup S\Gamma a]$ is a left Γ-ideal of S.
Proof Assume that $(S,\cdot,\leq)$ is an ordered Γ-AG-groupoid with left identity. By Lemma 2.3, we have $(a\cup S\Gamma a] = ((a\cup S\Gamma a]]$. Then

$$\begin{aligned} S\Gamma(a\cup S\Gamma a] &= (S]\Gamma(a\cup S\Gamma a] \\ &\subseteq (S\Gamma a\cup S\Gamma(S\Gamma a)] \\ &= (S\Gamma a\cup(a\Gamma S)\Gamma S] \\ &= (S\Gamma a\cup(S\Gamma S)\Gamma a] \\ &= (S\Gamma a\cup S\Gamma a] \\ &= (S\Gamma a] \\ &\subseteq (a\cup S\Gamma a]. \end{aligned}$$

Therefore $(a\cup S\Gamma a]$ is a left Γ-ideal of S.

Proposition 2.12. If $(S,\cdot,\leq)$ is an ordered Γ-AG-groupoid with left identity and let $a\in S,$ then $(S\Gamma a\cup a\Gamma S]$ is a Γ-ideal of $S.$

Proof Assume that $(S,\cdot,\leq)$ is an ordered Γ-AG-groupoid with left identity. By Lemma 2.3, we have $(a\Gamma S\cup a\Gamma S]=((a\Gamma S\cup a\Gamma S]].$ Then

$$\begin{aligned}(a\Gamma S\cup S\Gamma a]\Gamma S &= ((a\Gamma S\cup S\Gamma a)\Gamma S\,]\\ &= ((a\Gamma S\,)\Gamma S\cup(S\Gamma a)\Gamma S\,]\\ &= ((S\Gamma S\,)\Gamma a\cup S\Gamma(a\Gamma S\,)]\\ &= (S\Gamma a\cup a\Gamma(S\Gamma S\,)]\\ &= (S\Gamma a\cup a\Gamma S].\end{aligned}$$

Therefore $(S\Gamma a\cup a\Gamma S]$ is a right Γ-ideal of $S.$ By Lemma 2.5, we have $(S\Gamma a\cup a\Gamma S]$ is a Γ-ideal of $S.$

Lemma 2.13. If $(S,\cdot,\leq)$ is an ordered Γ-AG-groupoid with left identity and let $a\in S,$ then $(a\cup S\Gamma a\cup a\Gamma S\,]$ is a Γ-ideal of $S.$

Proof Assume that $(S,\cdot,\leq)$ is an ordered Γ-AG-groupoid with left identity. By Lemma 2.3, we have $(a\cup S\Gamma a\cup a\Gamma S]=((a\cup S\Gamma a\cup a\Gamma S]].$ Then

$$\begin{aligned}(a\cup S\Gamma a\cup a\Gamma S\,]\Gamma S &= (a\cup S\Gamma a\cup a\Gamma S]\Gamma(S\,]\\ &\subseteq (a\Gamma S\cup(S\Gamma a)\Gamma S\cup(a\Gamma S\,)\Gamma S]\\ &= (a\Gamma S\cup S\Gamma(a\Gamma S\,)\cup(S\Gamma S\,)\Gamma a]\\ &= (a\Gamma S\cup a\Gamma(S\Gamma S\,)\cup S\Gamma a]\\ &= (S\Gamma a\cup a\Gamma S\cup S\Gamma a]\\ &= (S\Gamma a\cup a\Gamma S]\\ &\subseteq (a\cup S\Gamma a\cup a\Gamma S\,].\end{aligned}$$

Therefore $(a\cup S\Gamma a\cup a\Gamma S\,]$ is a right Γ-ideal of $S.$ By Lemma 2.5, we have $(a\cup S\Gamma a\cup a\Gamma S\,]$ is a Γ-ideal of $S.$

Proposition 2.14. If $(S,\cdot,\leq)$ is an ordered Γ-AG-groupoid with left identity and let $a\in S,$ then $(a^2\cup a^2\Gamma S]$ is an Γ-ideal of $S.$

Proof Assume that $(S,\cdot,\leq)$ is an ordered Γ-AG-groupoid with left identity. By Lemma 2.3, we have $(a^2\cup a\Gamma S]=((a^2\cup a^2\Gamma S]].$ Then

$$\begin{aligned}(a^2\cup S\Gamma a^2\,]\Gamma S &= ((a^2\cup a^2\Gamma S)\Gamma S]\\ &= (a^2\Gamma S\cup(a^2\Gamma S\,)\Gamma S]\\ &= (a^2\Gamma S\cup a^2\Gamma(S\Gamma S\,)]\\ &= (a^2\Gamma S\cup a^2\Gamma S]\\ &\subseteq (a^2\cup a^2\Gamma S\cup a^2\cup a^2\Gamma S]\\ &= (a^2\cup a^2\Gamma S].\end{aligned}$$

Therefore $(a^2\cup a^2\Gamma S]$ is a right Γ-ideal of $S.$ By Lemma 2.5, we have $(a^2\cup a^2\Gamma S]$ is a Γ-ideal of $S.$

3. MAIN RESULTS

We start with the following theorem that gives a relation between left semiprime and quasi semiprime Γ-ideal in ordered Γ-AG-groupoid. Our starting points is the following definition:

Definition 3.1. Let $(S,\cdot,\leq)$ be an ordered Γ-AG-groupoid. A Γ-ideal P of R is called semiprime if every Γ-ideal A of S such that $A\Gamma A\subseteq P,$ we have $A\subseteq P.$ A left Γ-ideal P of S is called quasi-semiprime if every left Γ-ideal A of S such that $A\Gamma A\subseteq P,$ we have $A\subseteq P.$

Lemma 3.2. Let $(S,\cdot,\leq)$ be an ordered Γ-AG-groupoid with left identity. Then a Γ-ideal P of S is quasi-semiprime if and only if $a\gamma a\in P$ implies that $a\in P,$ where $a\in S$ and $\gamma\in\Gamma.$

Proof $\Rightarrow$ Assume that $(S,\cdot,\leq)$ is an ordered Γ-AG-groupoid with left identity. Then by hypothesis, $a\gamma a\in P$ for any $a\in S$ and $\gamma\in\Gamma.$ Then

$$\begin{aligned}
(a\cup S\Gamma a]\Gamma(a\cup S\Gamma a] &\subseteq ((a\cup S\Gamma a)\Gamma(a\cup S\Gamma a)]\\
&= (a^2\cup a\Gamma(S\Gamma a)\cup(S\Gamma a)\Gamma a\cup(S\Gamma a)\Gamma(S\Gamma a)]\\
&\subseteq (P\cup S\Gamma(a\Gamma a)\cup a^2\Gamma S\cup S\Gamma((S\Gamma a)\Gamma a)]\\
&\subseteq (P\cup S\Gamma P\cup P\Gamma S\cup S\Gamma(a^2\Gamma S)]\\
&= (P\cup P\cup P\cup S\Gamma(P\Gamma S)]\\
&\subseteq (P]\\
&= P.
\end{aligned}$$

By hypothesis, $a\in(a\cup S\Gamma a]$ and so that $a\in P.$

$\Leftarrow$ It is obvious.

Lemma 3.3. Let $(S,\cdot,\leq)$ be an ordered Γ-AG-groupoid with left identity. Then a Γ-ideal P of S is semiprime if and only if $a\gamma a\in P$ implies that $a\in P,$ where $a\in S$ and $\gamma\in\Gamma.$

Proof $\Rightarrow$ Assume that $(S,\cdot,\leq)$ is an ordered Γ-AG-groupoid with left identity. Then by hypothesis, $a\gamma a\in P$ for any $a\in S$ and $\gamma\in\Gamma.$ Then

$$\begin{aligned}
(a\cup S\Gamma a\cup a\Gamma S]\Gamma(a\cup S\Gamma a\cup a\Gamma S] &\subseteq ((a\cup S\Gamma a\cup a\Gamma S)\Gamma(a\cup S\Gamma a\cup a\Gamma S)]\\
&= (a\Gamma(a\cup S\Gamma a\cup a\Gamma S)\cup\\
&\quad (S\Gamma a)\Gamma(a\cup S\Gamma a\cup a\Gamma S)\cup\\
&\quad (a\Gamma S)\Gamma(a\cup S\Gamma a\cup a\Gamma S)]\\
&= (a\Gamma a\cup a\Gamma(S\Gamma a)\cup a\Gamma(a\Gamma S)\cup(S\Gamma a)\Gamma a\cup\\
&\quad (S\Gamma a)\Gamma(S\Gamma a)\cup(S\Gamma a)\Gamma(a\Gamma S)\cup(a\Gamma S)\Gamma a\cup\\
&\quad (a\Gamma S)\Gamma(S\Gamma)\cup(a\Gamma S)\Gamma(a\Gamma S)]\\
&\subseteq (P\cup(S\Gamma a)\Gamma(S\Gamma a)\cup(S\Gamma a)\Gamma(a\Gamma S)\cup\\
&\quad (S\Gamma a)\Gamma(S\Gamma a)\cup(S\Gamma a)\Gamma(S\Gamma a)\cup\\
&\quad (S\Gamma a)\Gamma(a\Gamma S)\cup(a\Gamma S)\Gamma(S\Gamma a)\cup\\
&\quad (a\Gamma S)\Gamma(S\Gamma a)\cup(a\Gamma S)\Gamma(a\Gamma S)]\\
&= (P\cup(S\Gamma a)\Gamma(S\Gamma a)\cup(S\Gamma a)\Gamma(a\Gamma S)\cup\\
&\quad (a\Gamma S)\Gamma(S\Gamma a)\cup(a\Gamma S)\Gamma(a\Gamma S)]\\
&= (P\cup((S\Gamma a)\Gamma a)\Gamma S\cup((a\Gamma S)\Gamma a)\Gamma S\cup
\end{aligned}$$

$$
\begin{aligned}
& S\Gamma((a\Gamma S)\Gamma a) \cup ((a\Gamma S)\Gamma S)\Gamma a] \\
= \;& (P \cup ((a\Gamma a)\Gamma S)\Gamma S \cup S\Gamma(a\Gamma(S\Gamma a)) \cup \\
& S\Gamma(a\Gamma(S\Gamma a)) \cup ((S\Gamma S)\Gamma a)\Gamma a] \\
= \;& (P \cup (P\Gamma S)\Gamma S \cup S\Gamma(S\Gamma(a\Gamma a)) \cup \\
& S\Gamma(S\Gamma(a\Gamma a)) \cup (a\Gamma a)\Gamma S] \\
\subseteq \;& (P \cup P \cup S\Gamma(S\Gamma P) \cup S\Gamma(S\Gamma P) \cup P\Gamma S] \\
\subseteq \;& (P \cup P \cup P \cup P \cup P] \\
\subseteq \;& (P] \\
= \;& P.
\end{aligned}
$$

By hypothesis, $a \in (a \cup S\Gamma a \cup a\Gamma S]$ and so that $a \in P.$
$\Leftarrow$ It is obvious.

Theorem 3.4. Let $(S,\cdot,\leq)$ be an ordered Γ-AG-groupoid with left identity. Then a Γ-ideal P of S is semiprime if and only if P is quasi-semiprime.
Proof This follows from Lemma 3.2 and Lemma 3.3.

Theorem 3.5. Let $(S,\cdot,\leq)$ be an ordered Γ-AG-groupoid with left identity and let P be a Γ-ideal of $S.$ Then a Γ-ideal P of S is semiprime if and only if $(a\Gamma(S\Gamma a)] \subseteq P$ implies that $a \in P,$ where $a \in S.$
Proof $\Rightarrow$ Assume that $(S,\cdot,\leq)$ is an ordered Γ-AG-groupoid with left identity. Then

$$
\begin{aligned}
a\gamma a \in (S\Gamma a)\Gamma(S\Gamma a) \;&=\; ((S\Gamma a)\Gamma a)\Gamma S \\
&=\; S\Gamma(a\Gamma(S\Gamma a)) \\
&=\; S\Gamma(a\Gamma(S\Gamma a)) \\
&=\; S\Gamma(a\Gamma(S\Gamma a))] \\
&\subseteq\; S\Gamma P \\
&\subseteq\; P.
\end{aligned}
$$

By Lemma 3.3, we have $a \in P.$
$\Leftarrow$ It is obvious.

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