On Left Primary and Weakly Left Primary Ideals in LA- Rings

Pairote Yiarayong[1]

[1]Faculty of Science and Technology,
Pibulsongkram Rajabhat University,
Phitsanuloke 65000, Thailand
E-mail: pairote0027 {at} hotmail.com

ABSTRACT— *In this paper, we study left ideals, left primary and weakly left primary ideals in LA-rings. Some characterizations of left primary and weakly left primary ideals are obtained. Moreover, we investigate relationships left primary and weakly left primary ideals in LA-rings. Finally, we obtain necessary and sufficient conditions of a weakly left primary ideal to be a left primary ideals in LA- rings.*

Keywords— LA-ring, left primary ideal, weakly left primary ideal, left (right) ideal.

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1. INTRODUCTION

A groupoid S is called an Abel-Grassmann's groupoid, abbreviated as an AG-groupoid, if its elements satisfy the left invertive law [1, 2], that is: for all Several examples and interesting properties of AG-groupoids can be found in [3, 4, 5] and [6]. It has been shown in [3] that if an AG-groupoid contains a left identity then it is unique. It has been proved also that an AG-groupoid with right identity is a commutative monoid, that is, a semigroup with identity element. It is also known [2] that in an AG-groupoid, the medial law, that is,

$$(ab)(cd) = (ac)(bd)$$

for all $a,b,bc,d \in S$ holds. Now we define the concepts that we will used. Let S be an AG-groupoid. By an AG-subgroupoid of [8], we means a non-empty subset A of S such that $A^2 \subseteq A.$ A non-empty subset A of an AG-groupoid S is called a left (right) ideal of [7] if $SA \subseteq A(AS \subseteq A).$ By two-sided ideal or simply ideal, we mean a non-empty subset of an AG-groupoid S which is both a left and a right ideal of $S.$

S.M. Yusuf in [20] introduces the concept of a left almost ring (LA-ring). That is, a non-empty set R with two binary operations “+” and “$\cdot$” is called a left almost ring, if $(R,+)$ is a LA-group, $(R,\cdot)$ is a LA-semigroup and distributive laws of “$\cdot$” over “+” holds. Further in [12] T. Shah and I. Rehman generalize the notions of commutative semigroup rings into LA-semigroup LA-rings. However T. Shah and Fazal ur Rehman in [12] generalize the notion of a LA-ring into a nLA-ring. A near left almost ring (nLA-ring) N is a LA-group under “+”, a LA-semigroup under “$\cdot$” and left distributive property of “$\cdot$” over “+” holds.

T. Shah, Fazal ur Rehman and M. Raees asserted that a commutative ring $(R,+,\cdot),$ we can always obtain a LA-ring $(R,\oplus,\cdot)$ by defining, for $a,b,c \in R, a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. Furthermore, in this paper we characterize the left primary and weakly left primary ideals in LA-rings. Moreover, we investigate relationships left primary and weakly left primary ideals in LA-rings. Finally, we obtain necessary and sufficient conditions of a weakly left primary ideal to be a left primary ideals in LA-rings.

2. IDEALS IN LA-RINGS

The results of the following lemmas seem play an important role to study LA-ring; these facts will be used so frequently that normally we shall make no reference to this lemma.

Definition 2.1. [11] A non empty set R with two binary operations “+” and “$\cdot$” is called a left almost ring if and only if

1. $(R,+)$ is a LA-group.
2. $(R,\cdot)$ is a LA-semigroup.
3. Left distributive property of “+” and “$\cdot$” holds.

Lemma 2.2. [14] Let $(R,+,\cdot)$ be a LA-ring, then for all $a,b,c\in R$

1. $0\cdot a=0=a\cdot 0.$
2. $a(-b) = -ab = (-a)b.$
3. $-(-a) = a.$
4. $(-a)(-b) = ab.$

Lemma 2.3. Let R be a LA-ring with left identity $e.$ Then $RR=R$ and $R=eR=Re.$
Proof. Let R be a LA-ring with left identity e and let $r\in R$ then $r=er\in RR,$ for all so that $R\subseteq RR.$ Since R is a LA-ring, we have $RR\subseteq R.$ Thus $RR=R.$ Now as e is a left identity in $R, ea=a,$ for all $a\in R.$ Then $R=eR.$ Since $(ab)c=(cb)a,$ for all $a,b,c\in R,$ we have $(RR)e=(eR)R.$ Now,

$$Re=(RR)e=(eR)R=RR=R.$$

Hence $R=eR=Re.$

Definition 2.4. [11] A nonempty subset I of a LA-ring R is a subring of R if under the binary operations in $R,$ form a LA-ring.

Definition 2.5. [11] A subring I of R is called a left (right) ideal of R if $RI\subseteq I(IR\subseteq I)$ and is called ideal if it is left as well as right ideal.

Lemma 2.6. If R is a LA-ring with left identity, then every right ideal is a left ideal.
Proof. Let R be a LA-ring with left identity and let A be a right ideal of $R.$ Then for $a\in A, r\in R$ consider

$$\begin{aligned} ra &= (er)a \\ &= (ar)e \\ &\in (AR)R \\ &\subseteq AR \\ &\subseteq A, \end{aligned}$$

where e is a left identity, that is $ra\in A.$ Therefore A is left ideal of $R.$

Lemma 2.7. If I is a left ideal of a LA-ring R with left identity, and if for any $a\in R,$ then aI is a left ideal of $R.$
Proof. Let I be a left ideal of $R,$ consider

$$\begin{aligned} s(ai) &= (es)(ai) \\ &= (ea)(si) \\ &= a(si) \\ &\in a(RI) \\ &\subseteq aI \end{aligned}$$

and $(ai)+(aj)=a(i+j)\in aI.$ Hence aI is a left ideal of $R.$

Lemma 2.8. Let R be a LA-ring with left identity, and $a\in R.$ Then Ra is a left ideal of $R.$
Proof. Let R be a LA-ring with left identity, and $a\in R.$ Then

$$\begin{aligned} R(Ra) &= (RR)(Ra) \\ &= (aR)(RR) \\ &= (aR)R \\ &= (RR)a \\ &= Ra \end{aligned}$$

and $(ra)+(sa)=(r+s)a\in Ra.$ Hence Ra is a left ideal of $R.$

Lemma 2.9. If I is an ideal of a LA-ring R with left identity, and if for any $a \in R,$ then $a^2 I$ is an ideal of $R.$

Proof. By Lemma 2.7, we have $a^2 I$ is a left ideal of $R.$ Now consider

$$\begin{aligned}
(a^2 r)s &= ((aa)r)s \\
&= ((ra)a)s \\
&= [e((ra)a)]s \\
&= [s((ra)a)]e \\
&= [(ra)(sa)]e \\
&= [((sa)a)r]e \\
&= [((aa)s)r]e \\
&= [(rs)(aa)]e \\
&= [e(aa)](rs) \\
&= (aa)(rs) \\
&= a^2(rs) \in a^2 I.
\end{aligned}$$

Hence $a^2 I$ is an ideal of $R.$

Lemma 2.10. Let R be a LA-ring with left identity, and $a \in R.$ Then Ra^2 is an ideal of $R.$

Proof. Let R be a LA-ring with left identity, and $a \in R.$ Now consider

$$\begin{aligned}
Ra^2 &= (RR)a^2 \\
&= a^2(RR) \\
&= a^2 R.
\end{aligned}$$

By Lemma 2.9, we have Ra^2 is an ideal of $R.$

Lemma 2.11. Let R be a LA-ring with left identity, and let A, B be left ideals of $R.$ Then $(A:B)$ is a left ideal in $R,$ where $(A:B) = \{r \in R : Br \subseteq A\}.$

Proof. Suppose that R is a LA-ring. Let $s \in R$ and let $a, b \in (A:B).$ Then $Ba \subseteq A$ and $Bb \subseteq A$ so that

$$\begin{aligned}
B(a+b) &= (Ba)+(Bb) \\
&\subseteq A+A \\
&= \text{A}
\end{aligned}$$

and

$$\begin{aligned}
B(sa) &= s(Ba) \\
&= sA \\
&= \text{A}.
\end{aligned}$$

Therefore $a+b \in (A:B)$ and $sa \in (A:B)$ so that $R(A:B) \subseteq (A:B).$ Hence $(A:B)$ is a left ideal in $R.$

Corollary 2.12. Let R be a LA-ring with left identity, and let A be left ideals of R Then $(A:b)$ is a left ideal in $R,$ where $(A:b) = \{r \in R : br \in A\}.$

Proof. This follows from Lemma 2.11.

Remark.1. Let R be a LA-ring and let A be a left ideal of $R.$ It is easy to verify that $A \subseteq (A:r).$

2. Let R be a LA-ring with left identity $e,$ and let A be a proper left (right) ideal of $R.$ By Corollary 2.12 , we have $e \notin (A:r),$ where $r \in R - A.$

2. Let R be a LA-ring and let A, B, C be left ideals of $R.$ It is easy to verify that $(A:C) \subseteq (A:B),$ where $B \subseteq C.$

3. LEFT PRIMARY AND WEAKLY LEFT PRIMARY IDEAL IN LA-RINGS

We start with the following theorem that gives a relation between left primary and weakly left primary ideal in Γ-LA-ring. Our starting points is the following definition:

Definition 3.1. A left ideal P is called left primary if $AB \subseteq P$ implies that $((AA)\ldots A)A = A^n \subseteq P$ or $B \subseteq P$ for some positive integer $n,$ where A, B is a left ideals of $R.$

Definition 3.2. A left ideal P is called weakly left primary if $0 \neq AB \subseteq P$ implies that $((AA)\ldots A)A = A^n \subseteq P$ or $B \subseteq P$ for some positive integer $n,$ where A, B is a left ideals of $R.$

Remark. It is easy to see that every left primary ideal is weakly left primary.

Lemma 3.3. If R is a LA-ring with left identity, then a left ideal P of R is left primary if and only if $ab \in P$ implies that $a^n \in P$ or $b \in P$ for some positive integer $n,$ where $a, b \in R.$

Proof. Let P be a left ideal of LA-ring R with left identity. Now suppose that $ab \in P.$ Then by Definition of left ideal, we get

$$\begin{aligned}(Ra)(Rb) &= (RR)(ab) \\ &= R(ab) \\ &\subseteq RP \\ &\subseteq P.\end{aligned}$$

Then $a^n \in P$ or $b \in P$ for some positive integer $n.$ Conversely, the proof is easy.

Corollary **3.4.** If R is a LA-ring with left identity, then a left ideal P of R is weakly left primary if and only if $0 \neq ab \in P$ implies that $a^n \in P$ or $b \in P$ for some positive integer $n,$ where $a, b \in R.$

Proof. This follows from Lemma 3.3.

Let R be a LA-ring and A be a subset of N. We write

$$\sqrt{A} = \{a \in N : a^k \in A \text{ for some positive integer } k\}.$$

Theorem 3.5. Let R be a LA-ring, and let P be an ideal of $R.$ If P is a weakly left primary ideal that is not let primary. Then $\sqrt{P} = \sqrt{0}.$

Proof. Let R be a LA-ring with identity. First, we prove that $P^2 = 0.$ Suppose that $P^2 \neq 0$ we show that P is weakly left primary. Let $ab \in P$, where $a, b \in R.$ If $ab \neq 0,$ then either

$$a \in \sqrt{P} \text{ or } b \in P$$

since P is weakly left primary ideal. So suppose that $ab = 0.$ If $Pb \neq 0,$ then there is an element p' of P such that $p'b \neq 0,$ so that

$$0 \neq p'b = (p' + a)b \in P,$$

and hence P weakly left primary ideal gives either $p' + a \in \sqrt{P}$ or $b \in P.$ As $p' + a \in \sqrt{P}$ and $p' \in P \subseteq \sqrt{P}$ we have either $a \in \sqrt{P}$ or $b \in P.$ So we can assume that $Pb = 0.$ Similarly, we can assume that $Pa = 0.$ Since $P^2 \neq 0,$ there exist $c, d \in P$ such that $cd \neq 0.$ Then

$$0 \neq (a + c)(b + d) \in P,$$

so either $a + c \in \sqrt{P}$ or $b + d \in P,$ and hence either $a \in \sqrt{P}$ or $b \in P.$ Thus P is left primary ideal. Clearly, $\sqrt{0} \subseteq \sqrt{P}.$ As $P^2 = 0,$ we get $\sqrt{P} \subseteq \sqrt{0},$ hence $\sqrt{P} = \sqrt{0},$ as required.

Corollary 3.6. Let R be a Γ-LA-ring, and let P an ideal of R. If $\sqrt{P} \neq \sqrt{0}$, then P is left primary if and only if P is weakly left primary.
Proof. This follows from Theorem 3.5.

Lemma 3.7. Let R be a LA-ring with identity, and let P be a proper ideal of R. If P is a weakly left primary ideal of R, then $(P:Ra) = P \cup (0:Ra)$, where $a \in R - \sqrt{P}$.
Proof. Let R be a LA-ring with identity, and let P be a weakly left primary ideal of R. Clearly,

$$P \cup (0:Ra) \subseteq (P:Ra).$$

For the other inclusion, suppose that $m \in (P:Ra)$, so that

$$\begin{aligned} (Ra)(Rm) &= (mR)(aR) \\ &= (ma)(RR) \\ &= (ma)R \\ &= (Ra)m \\ &\subseteq P. \end{aligned}$$

If $0 \neq (Ra)m$, then $m = em \in Rm \subseteq P$ since P is weakly left primary. If $0 = (Ra)m$, then $m \in (0:Ra)$ so we have the equality.

Corollary 3.8. Let R be a LA-ring with identity, and let P be a proper ideal of R. If P is a weakly left primary ideal of R, then $(P:a) = P \cup (0:a)$, where $a \in R - \sqrt{P}$.
Proof. This follows from Lemma 3.7.

Corollary 3.9. Let R be a LA-ring with identity, and let P be a proper ideal of R. If $(P:Ra) = P \cup (0:Ra)$, then $(P:Ra) = P$ or $(P:Ra) = (0:Ra)$, where $a \in R - \sqrt{P}$.
Proof. This follows from Lemma 3.7.

Theorem 3.10. Let R be a LA-ring with identity, and let P be a proper ideal of R. If $(P:n) = P$ or $(P:n) = (0:n)$, then P is a weakly left primary ideal of R, where $n \in R - \sqrt{P}$.
Proof. Let R be a LA-ring with identity, and let P be a proper ideal of R. Suppose that Let $0 \neq mn \in P$, where $m \in R - \sqrt{P}$. Then

$$m \in (P:n) = P \cup (0:n)$$

by Corollary 3.9 hence $m \in P$ since $mn \neq 0$, as required.

Lemma 3.11. Let $R = R_1 \times R_2$, where each R_i is a LA-ring with identity. Then the following hold:

(i) If A is a left ideal of R_1, then $\sqrt{A \times R_2} = \sqrt{A} \times R_2$.

(ii) If A is a left ideal of R_2, then $\sqrt{R_1 \times A} = R_1 \times \sqrt{A}$.

Proof. The proof is straightforward.

Theorem 3.12. Let $R = R_1 \times R_2$, where each R_i is a LA-ring with identity. If P is a weakly left primary (left primary) ideal of R_1, then $P \times R_2$ is a weakly left primary (left primary) ideal of R.
Proof. Suppose that $R = R_1 \times R_2$, where each R_i is a LA-ring with identity and P is a weakly left primary ideal of R_1. Let

$$0 \neq (a,b)(c,d) = (ac,bd) \in P \times R_2,$$

where $(a,b),(c,d) \in R$ so either $a \in \sqrt{P}$ *or* $c \in P$ since P is weakly left primary. It follows that either

$$(a,b) \in \sqrt{P} \times R_2 = \sqrt{P \times R_2} \text{ or } (c,d) \in P \times R_2.$$

By Definition of weakly left primary ideal, we have $P \times R_2$ is a weakly left primary ideal of R.

Corollary 3.13. Let $R = R_1 \times R_2$, where each R_i is a LA-ring with identity. If P is a weakly left primary (left primary) ideal of R_2, then $R_1 \times P$ is a weakly left primary (left primary) ideal of R.
Proof. This follows from Theorem 3.12.

Corollary 3.14. Let $R = \prod_{i=1}^{n} R_i$, where each R_i is a LA-ring with identity. If P is a weakly left primary (left primary) ideal of R_j, then $R_1 \times R_2 \times \ldots \times P_j \times R_{j+1} \times \ldots \times R_n$ is a weakly left primary (left primary) ideal of R.
Proof. This follows from Theorem 3.12 and Corollary 3.13.

Theorem 3.15. Let $R = R_1 \times R_2$, where each R_i is a LA-ring with identity. If P is a weakly left primary ideal of R, then either $P = 0$ or P is left primary.
Proof. Let $R = R_1 \times R_2$, where each R_i is a LA-ring with identity and let $P = R_1 \times P_2$ be a weakly left primary ideal of R. We can assume that $P \neq 0$. So there is an element (a,b) of P with $(a,b) \neq (0,0)$. Then

$$(0,0) \neq (a,e)(e,\ b) \in P,$$

gives either

$$(a,\ e) \in \sqrt{P} = \sqrt{P_1} \times R_2 \text{ or } (e,\ b) \in P$$

If $(e,\ b) \in P$, then $P = R_1 \times P_2$. We show that P_2 is left primary hence P is weakly left primary by Corollary 3.13. Let $cd \in P_2$, where $c,d \in R_2$. Then

$$(0,0) \neq (e,c)(e,d) = (e,cd) \in P,$$

so either $(e,c) \in \sqrt{P} = \sqrt{R_1 \times P_2} = R_1 \times \sqrt{P_2}$ or $(e,d) \in P$ and hence either $c \in \sqrt{P_2}$ or $d \in P_2$. By a similar argument, $P = R_1 \times P_2$ is left primary.

Proposition 3.16. Let $A \subseteq P$ be proper ideals of a LA-ring R. Then the following hold:

(i) If P is weakly left primary (left primary), then P/A is weakly left primary (left primary).

(ii) If A and P/A are weakly left primary (left primary), then P is weakly left primary (left primary).

Proof. (i) Let $0 \neq (a+A)(b+A) = ab + A \in P/A$, where $a,b \in R$ so $ab \in P$. If $ab = 0 \in A$, then

$$(a+A)(b+A) = 0,$$

a contradiction. So if P is weakly left primary, then either $a \in \sqrt{P}$ or $b \in P$, hence either $a + A \in \sqrt{P/A}$ or $b + A \in P/A$, as required.

(ii) Let $0 \neq ab \in P$, where $a,b \in R$, so $(a+A)(b+A) \in P/A$. For $ab \in A$, if A is weakly left primary, then either

$$a \in A \subseteq \sqrt{P} \text{ or } b \in A \subseteq P.$$

So we may assume that $ab \notin A$. Then either $a + A \in \sqrt{P/A}$ or $b + A \in P/A$. It follows that either $a \in \sqrt{P}$ or $b \in P$ as needed.

Theorem 3.17. Let P and Q be weakly left primary ideals of a LA -ring R that are not left primary. Then $P+Q$ is a weakly left primary ideal of R.

Proof. Since $(P+Q)/Q \approx Q/(P \cap Q)$, we get that $(P+Q)/Q$ is weakly left primary by Proposition 3.16 (i). Now the assertion follows from Proposition 3.16 (ii).

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