# A Method for Reduction of Spurious or Numerical Oscillations in Integration of Unsteady Boundary Value Problem

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**ABSTRACT—** ***Elimination of the spurious or numerical oscillations is very important in the solution of unsteady boundary value problem by FDM. Upwind differencing in advection problem is very popular, but numerical diffusion is too big. Flux limiters are very effective to eliminate the numerical oscillations, but the procedure is rather complicated. In the present paper, a very simple and unique method is proposed to reduce numerical oscillations. The method is verified by numerical calculations. This solution can be applied to many problems and to other solutions such as FEM, BEM etc. This solution can be applied not only to explicit method but also to implicit method. This solution can be extended easily to multi-dimensional problems.***

**Keywords—** spurious or numerical oscillation, Central differencing, Diffuser by moving average, Burgers' equation

## 1. INTRODUCTION

In solution of unsteady boundary value problems, the numerical instability and the numerical or spurious oscillation must be avoided. The former was discussed also by the present author [1] recently. The implicit solution is very effective to avoid it. On the other hand, Lax-Wendroff [2] and the flux limiter [3-5] methods are useful for reducing and eliminating the latter.

Elimination of the numerical oscillation is very important in FDM. Upwind differencing in advection problem is very popular. However, the upwind differencing generates rather large numerical diffusion.

In the present paper, a very simple and unique method for reducing the numerical oscillation is proposed. In many cases, computational noise of sinusoidal nature is generated. If a signal of opposite phase is generated by differentiating the original signal twice and is added to the original signal, this kind of noise can be cancelled easily. On the other hand, the rapidly varying component in the original signal can be made by subtracting the slowly varying component from the original signal, and the slowly varying component can be extracted by taking the moving average of the original signal. Hence, if we regard that the rapidly varying component in the original signal is the numerical oscillation, the numerical oscillation may be easily reduced or eliminated by subtracting the rapidly varying component from the original signal. This suggests us that taking the second derivative is equivalent to generating the rapidly varying component by averaging process.

The method is verified by numerical calculations, one- and two-dimensional examples are shown. This solution can be applied to many problems and to other solutions such as FEM, BEM etc.

## 2. SUMMARY OF EXISTING METHODS TO REDUCE SPURIOUS OR NUMERICAL OSCILLATIONS

Let us consider an initial value problem of advection equation:

$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}=0\,, \tag{1}$$

$$u(x,0)=f(x)\,. \tag{2}$$

In the following, we assume $U>0$ for simplicity.

The upwind difference is given by

$$u_i^{(n+1)}\equiv u_i^{(n)}-\frac{Udt}{dx}(u_i^{(n)}-u_{i-1}^{(n)})=u_i^{(n)}-\frac{Udt}{2dx}(u_{i+1}^{(n)}-u_{i-1}^{(n)})+\frac{Udxdt}{2}\frac{1}{dx^2}(u_{i+1}^{(n)}-2u_i^{(n)}+u_{i-1}^{(n)})\,. \tag{3}$$

Hence, the upwind difference is "central difference + diffusion with diffusion constant $\nu=Udx/2$ ". **The second**

**term on the right hand side is a source of spurious oscillation and the third term is the diffuser with diffusion const. $\nu = Udx/2$. However, the diffuser is excessive.** If we use the upwind differencing, a rather big numerical diffusion or spurious diffusion takes place. This is the drawback of the upwind differencing.

On the other hand, the central difference is defined as

$$u_i^{(n+1)} \equiv u_i^{(n)} - \frac{Udt}{2dx}(u_{i+1}^{(n)} - u_{i-1}^{(n)}) . \tag{4}$$

If we use the central differencing, a numerical or spurious oscillation occurs. This is the drawback of the central differencing. However, in case of the central differencing, if we decrease $dx$, the accuracy is increased, and the numerical oscillation is reduced. The reduction of $dx$ increases the computational cost.

If we rewrite Eq. (4), we have

$$u_i^{(n+1)} \equiv u_i^{(n)} - \frac{Udt}{2dx}(u_{i+1}^{(n)} - u_{i-1}^{(n)}) = u_i^{(n)} - \frac{Udt}{2dx}(u_{i+1}^{(n)} - u_i^{(n)}) - \frac{Udt}{2dx}(u_i^{(n)} - u_{i-1}^{(n)}) \tag{5}$$

and

$$u_i^{(n+1)} \equiv u_i^{(n)} - \frac{Udt}{2dx}(u_{i+1}^{(n)} - u_{i-1}^{(n)}) = u_i^{(n)} - \frac{Udt}{dx}(u_i^{(n)} - u_{i-1}^{(n)}) - \frac{Udtdx}{2}\frac{1}{dx^2}(u_{i+1}^{(n)} - 2u_i^{(n)} + u_{i-1}^{(n)}) . \tag{6}$$

Equations (5) gives a relationship between "central difference" and "forward plus backward difference", and Eq. (6) a relationship between "central difference" and "upwind difference". **In Eq. (6), the second term on RHS is upwind derivative. The upwind derivative generates an excessive diffusion. The third term on RHS is anti-diffuser. The anti-diffuser is too strong.** From Eq. (6), we know that the central difference corresponds to the upwind difference with negative diffusion coefficient $\nu = -Udx/2$.

Lax-Wendroff method [2-8] is an effective way to reduce the numerical oscillation and is obtained as follows. First, we have Taylor expansion:

$$u(x,t+dt) = u(x,t) + u_t(x,t)dt + \frac{1}{2}u_{tt}(x,t)dt^2 + \cdots . \tag{8}$$

Then, from Eq. (1), we notice

$$u_t = -Uu_x ,\ u_{tt} = (u_t)_t = -U(u_x)_t = U^2 u_{xx} ,\ \cdots . \tag{9}$$

Substituting Eq. (9) into Eq. (8), we derive

$$u(x,t+dt) = u(x,t) - Uu_x(x,t)dt + \frac{1}{2}U^2 u_{xx}(x,t)dt^2 + \cdots . \tag{10}$$

If we replace both $u_x(x,t)$ and $u_{xx}(x,t)$ in Eq. (10) by the central differences, we obtain, from Eq. (10)

$$u_i^{(n+1)} = u_i^{(n)} - \frac{Udt}{2dx}(u_{i+1}^{(n)} - u_{i-1}^{(n)}) + \frac{1}{2}U^2 dt^2 \frac{1}{dx^2}(u_{i+1}^{(n)} - 2u_i^{(n)} + u_{i-1}^{(n)}) . \tag{11}$$

**In Eq. (11), the second term on RHS is a source of spurious oscillation, and the third term is anti-diffuser with a strength improved more than upwind method, Eq. (3).** Equation (11) corresponds to convective diffusion with positive diffusion coefficient $\nu = U^2 dt/2$:

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = \frac{1}{2}U^2 dt \frac{\partial^2 u}{\partial x^2} . \tag{12}$$

The flux term or the diffusion term in Eq. (12) contributes to suppress the numerical oscillation due to the advection term. Hence, Eq. (11) gives better numerical solution than Eq. (4). Namely, Equation (11) improves Eq. (4).

According to [4], we rewrite Eq. (11) as follows. Substituting

$$\frac{1}{2}(u_{i+1}^{(n)} - u_{i-1}^{(n)}) = (u_i^{(n)} - u_{i-1}^{(n)}) + \frac{1}{2}(u_{i+1}^{(n)} - 2u_i^{(n)} + u_{i-1}^{(n)}) \tag{13}$$

into Eq. (11), we obtain

$$u_i^{(n+1)} = u_i^{(n)} - \frac{Udt}{dx}(u_i^{(n)} - u_{i-1}^{(n)}) - \frac{1}{2}\left(1 - \frac{Udt}{dx}\right)\frac{Udt}{dx}(u_{i+1}^{(n)} - 2u_i^{(n)} + u_{i-1}^{(n)}) . \tag{14}$$

The second term on the right hand side has a role of anti-diffuser.

Rewriting $u_{i+1}^{(n)} - 2u_i^{(n)} + u_{i-1}^{(n)}$ as

$$u_{i+1}^{(n)} - 2u_i^{(n)} + u_{i-1}^{(n)} = (u_{i+1}^{(n)} - u_i^{(n)}) - (u_i^{(n)} - u_{i-1}^{(n)}) = \Delta_-(u_{i+1}^{(n)} - u_i^{(n)}) = \Delta_-\Delta_-(u_{i+1}^{(n)}) , \tag{15}$$

where

$$\Delta_-(y_m) = y_m - y_{m-1} . \tag{16}$$

Substituting Eq. (15) into Eq. (14), we have

$$u_i^{(n+1)} = u_i^{(n)} - \frac{Udt}{dx}\Delta_-(u_i^{(n)}) - \frac{1}{2}\left(1-\frac{Udt}{dx}\right)\frac{Udt}{dx}\Delta_-\Delta_-(u_{i+1}^{(n)})\,. \tag{17}$$

Adjusting the anti-diffuser, we assume

$$u_i^{(n+1)} = u_i^{(n)} - \frac{Udt}{dx}\Delta_-(u_i^{(n)}) - \frac{1}{2}\left(1-\frac{Udt}{dx}\right)\frac{Udt}{dx}\Delta_-\left(\varphi\left(\frac{\Delta_-(u_i^{(n)})}{\Delta_-(u_{i+1}^{(n)})}\right)\Delta_-(u_{i+1}^{(n)})\right). \tag{18}$$

Equations (17) and (18) correspond to Eqs. (3.2) and (3.6) in [4], respectively.

Now, we introduce TV (Total Variation) $\mathrm{TV}(u^{(n)})$ [3]:

$$\mathrm{TV}(u^{(n)}) = \sum_i \left|u_{i+1}^{(n)} - u_i^{(n)}\right|. \tag{19}$$

We require TVD (Total Variation Diminishing):

$$\mathrm{TV}(u^{(n+1)}) \le \mathrm{TV}(u^{(n)})\,. \tag{20}$$

When

$$u_i^{(n+1)} = u_i^{(n)} - C_i\Delta_-(u_i^{(n)}) + D_{i+1}\Delta_-(u_{i+1}^{(n)})\,, \tag{21}$$

then, the condition to satisfy TVD condition, Eq. (20), is given from [3, 4] by

$$0 \le C_i\,,\ 0 \le D_i\,,\ 0 \le C_i + D_i \le 1\,. \tag{22}$$

Applying Eq. (22) to Eq. (18), we obtain

$$C_i = \frac{Udt}{dx}\left[1+\frac{1}{2}\left(1-\frac{Udt}{dx}\right)\Delta_-\left(\phi\left(\frac{\Delta_-(u_i^{(n)})}{\Delta_-(u_{i+1}^{(n)})}\right)\Delta_-(u_{i+1}^{(n)})\right)\Big/\Delta_-(u_i^{(n)})\right],\ D_i = 0\,. \tag{23}$$

since

$$u_i^{(n+1)} = u_i^{(n)} - \frac{Udt}{dx}\left[1+\frac{1}{2}\left(1-\frac{Udt}{dx}\right)\Delta_-\left(\phi\left(\frac{\Delta_-(u_i^{(n)})}{\Delta_-(u_{i+1}^{(n)})}\right)\Delta_-(u_{i+1}^{(n)})\right)\Big/\Delta_-(u_i^{(n)})\right]\Delta_-(u_i^{(n)})\,. \tag{24}$$

Rewriting Eq. (23), we have

$$\begin{aligned} C_i &= \frac{Udt}{dx}\left[1+\frac{1}{2}\left(1-\frac{Udt}{dx}\right)\left(\phi\left(\frac{\Delta_-(u_i^{(n)})}{\Delta_-(u_{i+1}^{(n)})}\right)\Delta_-(u_{i+1}^{(n)}) - \phi\left(\frac{\Delta_-(u_{i-1}^{(n)})}{\Delta_-(u_i^{(n)})}\right)\Delta_-(u_i^{(n)})\right)\Big/\Delta_-(u_i^{(n)})\right] \\ &= \frac{Udt}{dx}\left[1+\frac{1}{2}\left(1-\frac{Udt}{dx}\right)\left(\phi\left(\frac{\Delta_-(u_i^{(n)})}{\Delta_-(u_{i+1}^{(n)})}\right)\frac{\Delta_-(u_{i+1}^{(n)})}{\Delta_-(u_i^{(n)})} - \phi\left(\frac{\Delta_-(u_{i-1}^{(n)})}{\Delta_-(u_i^{(n)})}\right)\right)\right]. \end{aligned} \tag{25}$$

Equations (24) and (25) correspond to Eqs. (3.8) and (3.9) in [4], respectively.

From Eqs. (22) and (23), we have $0 \le C_i \le 1$. If we define $\Phi$ as

$$\left|\phi\left(\frac{\Delta_-(u_i^{(n)})}{\Delta_-(u_{i+1}^{(n)})}\right)\frac{\Delta_-(u_{i+1}^{(n)})}{\Delta_-(u_i^{(n)})} - \phi\left(\frac{\Delta_-(u_{i-1}^{(n)})}{\Delta_-(u_i^{(n)})}\right)\right| \le \Phi\,, \tag{26}$$

then, we obtain

$$\frac{Udt}{dx}\left[1-\frac{1}{2}\left(1-\frac{Udt}{dx}\right)\Phi\right] \le C_i \le \frac{Udt}{dx}\left[1+\frac{1}{2}\left(1-\frac{Udt}{dx}\right)\Phi\right], \tag{27}$$

where Courant number is assumed to satisfy

$$\frac{Udt}{dx} < 1\,. \tag{27}$$

From the left inequality in Eq. (27), we have that

$$\Phi \le 2\,. \tag{28}$$

is sufficient to satisfy Eq. (22). If we require to maximize the role of anti-diffuser, we may asuume

$$\phi \ge 0 \ \text{ and } \ \phi = 0 \ \text{ when } \ r_i = \frac{\Delta_-(u_i^{(n)})}{\Delta_-(u_{i+1}^{(n)})} = \frac{u_i^{(n)} - u_{i-1}^{(n)}}{u_{i+1}^{(n)} - u_i^{(n)}} \le 0\,. \tag{29}$$

Then, we obtain that

$$0 \le \phi(r_i) = \phi\left(\frac{u_i^{(n)} - u_{i-1}^{(n)}}{u_{i+1}^{(n)} - u_i^{(n)}}\right) \le 2 \ \text{ and } \ 0 \le \frac{\phi(r_i)}{r_i} = \phi\left(\frac{u_i^{(n)} - u_{i-1}^{(n)}}{u_{i+1}^{(n)} - u_i^{(n)}}\right)\frac{u_{i+1}^{(n)} - u_i^{(n)}}{u_i^{(n)} - u_{i-1}^{(n)}} \le 2 \tag{30}$$

is sufficient to satisfy Eqs. (26) and (28). Hence, $\phi(r)$ must be in a region of Figure 1(a) surrounded by $\varphi = 2r$, $\varphi = 2$ and $\varphi = 0$. In order to maximize the role of anti-diffuser, $\phi(r)$ should be the maximum, that is

$$\phi(r) = \min(2r, 2), \tag{31}$$

which is the upper boundary of the region [4]. The region ①+② in Figure 1(b) gives the second order TVD region [4].

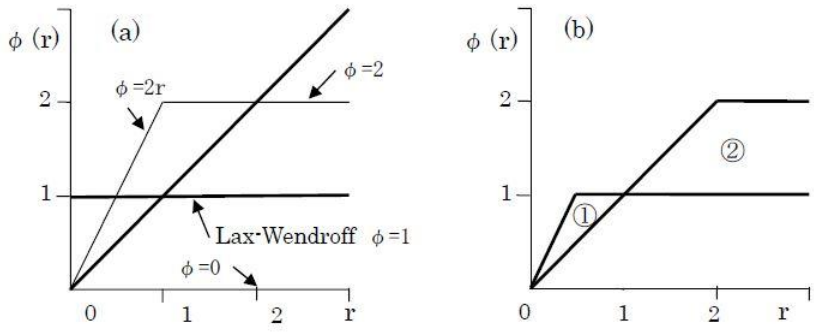


**Figure 1:** TVD regions [4]

More detailed specification of explanation of choosing $\phi(r)$ is given in [3] and [5]. Fifteen limiter functions are introduced in [5]. For example, "superbee" and "osprey" limiters are shown in Figure 2. Superbee-symmetric limiter of Roe is given by

$$\phi(r) = \max[0, \min(2r, 1), \min(r, 2)], \ \lim_{r\to\infty} \phi(r) = 2, \tag{32}$$

and Ospre-symmetric limiter of Waterson and Deconinck by

$$\phi(r) = \frac{1.5(r^2 + r)}{r^2 + r + 1}, \ \lim_{r\to\infty} \phi(r) = 1.5. \tag{33}$$

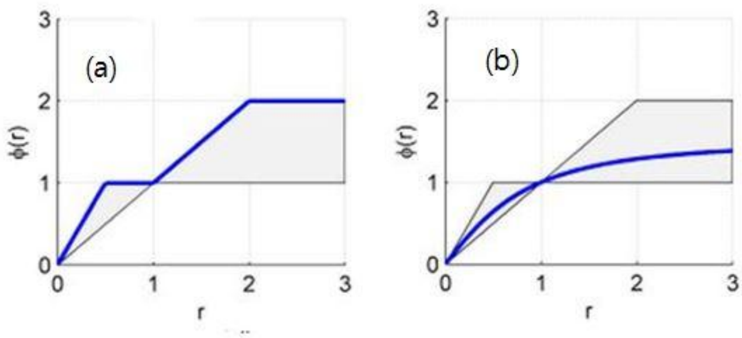


**Figure 2:** (a) Superbee limiter; (b) Ospre limiter [5]

Previous efforts by many researchers to eliminate and minimize the spurious oscillation is well explained in reference [5]. There are remarks in [5]: **"The various limiters have differing switching characteristics and are selected according to the particular problem and solution scheme. No particular limiter has been found to work well for all problems, and a particular choice is usually made on a trial and error basis."**

Similar results would also be obtained by choosing proper values of parameters $\alpha_i$ and $\beta_i$ in the following procedures:

$$u_i^{(n+1)} = u_i^{(n)} - \frac{Udt}{2dx}(u_{i+1}^{(n)} - u_{i-1}^{(n)}) + \alpha_i \frac{1}{2}\left(\frac{Udt}{dx}\right)^2 (u_{i+1}^{(n)} - 2u_i^{(n)} + u_{i-1}^{(n)}), \ \alpha > 0 \tag{34}$$

or

$$u_i^{(n+1)} = u_i^{(n)} - \frac{Udt}{dx}(u_i^{(n)} - u_{i-1}^{(n)}) - \beta_i \frac{1}{2}\left(1 - \frac{Udt}{dx}\right)\frac{Udt}{dx}(u_{i+1}^{(n)} - 2u_i^{(n)} + u_{i-1}^{(n)}), \ \beta > 0. \tag{35}$$

Even if we use constant values for $\alpha_i$ and $\beta_i$, these procedures would give useful results at moderate costs.

## 3. REDUCTION OF NUMERICAL OSCILLATIONS BY MOVING AVERAGE

Let $dx$ be the space interval of the adjacent signal $y_i = y(idx)$, $i = 0, 1, \cdots$. The rapid variation $F(y)$ of the signal is given by subtracting the moving average from the signal:

$$F_2(y_i) = -y_i + \frac{1}{2}(y_{i-1} + y_{i+1}), \tag{36a}$$

$$F_3(y_i) = -y_i + \frac{1}{3}(y_{i-1} + y_i + y_{i+1}), \tag{36b}$$

$$F_5(y_i) = -y_i + \frac{1}{5}(y_{i-2} + y_{i-1} + y_i + y_{i+1} + y_{i+2}), \tag{36c}$$

$$F_7(y_i) = -y_i + \frac{1}{7}(y_{i-3} + y_{i-2} + y_{i-1} + y_i + y_{i+1} + y_{i+2} + y_{i+3}). \tag{36d}$$

On the other hand, we have

$$F_2(y_i) = \frac{1}{2}(y_{i-1} - 2y_i + y_{i+1}) \approx \frac{dx^2}{2} y''(y_i), \tag{37a}$$

$$F_3(y_i) = \frac{1}{3}(y_{i-1} - 2y_i + y_{i+1}) \approx \frac{dx^2}{3} y''(y_i). \tag{37b}$$

If we consider the rapid variation corresponds to the computational noise, the noise is reduced in a signal $G(y)$ as shown in Figure 3, where

$$G(y) = y + F(y). \tag{38}$$

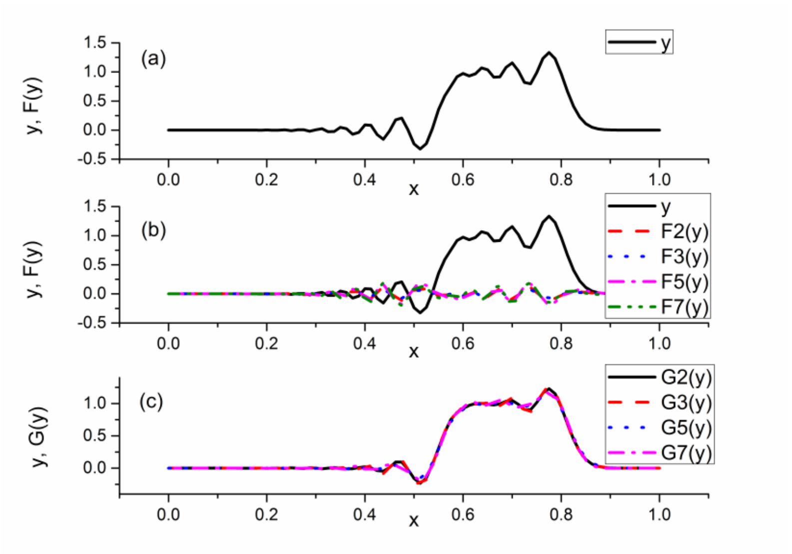


**Figure 3:** Reduction of noise from a signal ((a) Signal with noise; (b) (High-pass signal)×(-1); (c) (Signal with noise) - (High-pass signal))

If we repeat this process, the noise is reduced further as shown in Figure 4.

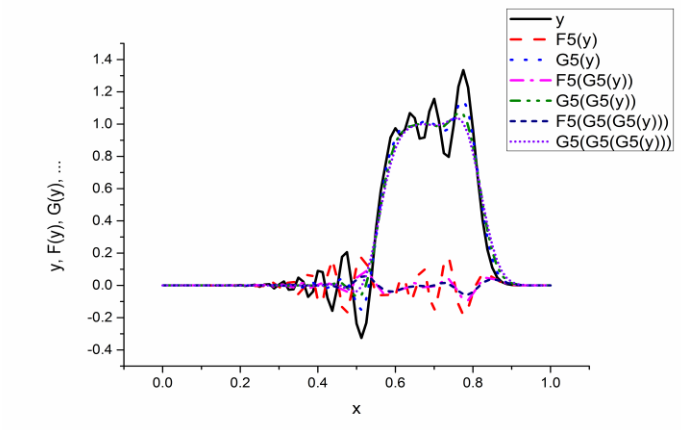


**Figure 4:** Reduction of noise from a signal

## 4. APPLICATION TO ONE_DIMENSIONAL (1D) BURGERS' EQUATION

The 1D Burger's equation is given by

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right)+f \quad \text{in} \quad -\infty<x<\infty \tag{39}$$

or

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}+\frac{\partial \nu}{\partial x}\frac{\partial u}{\partial x}+f \quad \text{in} \quad -\infty<x<\infty . \tag{40}$$

In case

$$\nu=const \quad \text{and} \quad f=0 \ \text{in} \ -\infty<x<\infty , \tag{41, 42}$$

we have

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2} \quad \text{in} \quad -\infty<x<\infty . \tag{43}$$

We approximate the infinite region by a finite one $-L<x<L$ and discretize the finite region as

$$dx=\frac{2L}{M}, \quad x_i=-L+i\,dx, \quad i=0,1,\cdots,M , \tag{44a, b}$$

$$u_i^{(n)}=u(x_i+ndt) . \tag{45}$$

The central differencing equation is given by

$$\frac{u_i^{(n+1)}-u_i^{(n)}}{dt}+u_i^{(n)}\frac{u_{i+1}^{(n)}-u_{i-1}^{(n)}}{2dx}=\nu\frac{u_{i+1}^{(n)}-2u_i^{(n)}+u_{i-1}^{(n)}}{dx^2} \quad \text{for } i=1,2,\cdots,M . \tag{46}$$

In case of the upwind differencing, the derivative in the advection term or the second term on the left-hand side of equation (43) is replaced by the upwind derivatives. Namely

$$\frac{u_{i+1}^{(n)}-u_{i-1}^{(n)}}{2dx}\rightarrow\begin{cases}\dfrac{u_i^{(n)}-u_{i-1}^{(n)}}{dx} & \text{if } u_i^{(n)}>0\\[2mm] \dfrac{u_{i+1}^{(n)}-u_i^{(n)}}{dx} & \text{if } u_i^{(n)}<0 .\\[2mm] \dfrac{u_{i+1}^{(n)}-u_{i-1}^{(n)}}{2dx} & \text{otherwise}\end{cases} \tag{47}$$

If we use Euler Solution, $u_i^{(n+1)}$ is calculated from $u_i^{(n)}$ explicitly.

The initial condition is given by

$$u_i^{(0)}=\begin{cases}U+4/(3L) & \text{when } -L/2\le x_i\le L/4\\ 0 & \text{otherwise}\end{cases} . \tag{48}$$

The exact solution of this problem [9] is given by

$$\begin{aligned}u&=-2\nu\frac{\partial}{\partial x}\ln\left[\frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-\xi)^2}{4\nu t}-\frac{1}{2\nu}\int_0^{\xi}u(\eta,0)\,d\eta\right)d\xi\right]\\ &=-2\nu\left[\frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-\xi)^2}{4\nu t}-\frac{1}{2\nu}\int_0^{\xi}u(\eta,0)\,d\eta\right)d\xi\right]^{-1}\\ &\quad\cdot\frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}-\frac{(x-\xi)}{2\nu t}\exp\left(-\frac{(x-\xi)^2}{4\nu t}-\frac{1}{2\nu}\int_0^{\xi}u(\eta,0)\,d\eta\right)d\xi .\end{aligned} \tag{49}$$

In order to reduce the numerical oscillations, after $u_i^{(n+1)}$ is calculated from $u_i^{(n)}$ by Eq. (46) at the end of time step $n$, $u_i^{(n+1)}$ is modified by

$$u_i^{(n+1)}-\alpha\left[u_i^{(n+1)}-\frac{1}{2K+1}\left(u_{i+K}^{(n+1)}+\cdots+u_{i+1}^{(n+1)}+u_i^{(n+1)}+u_{i-1}^{(n+1)}+\cdots+u_{i-K}^{(n+1)}\right)\right]\rightarrow u_i^{(n+1)} , \tag{50}$$

where the quantity in the square bracket is the rapidly varying component in the signal $u_i^{(n+1)}$ generated by subtracting slowly varying component or the moving average from the signal $u_i^{(n+1)}$.

**Table 1:** Computational condition

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| *L* | 6 | *M* | 21, 41, 81, 161, 321 | $\nu$ | 0.01 | *dt* | 0.00025 |
| *U* | 1 | *K* | 1 | $\alpha$ | 0, 0.001, 0.0015, 0.002 | | |

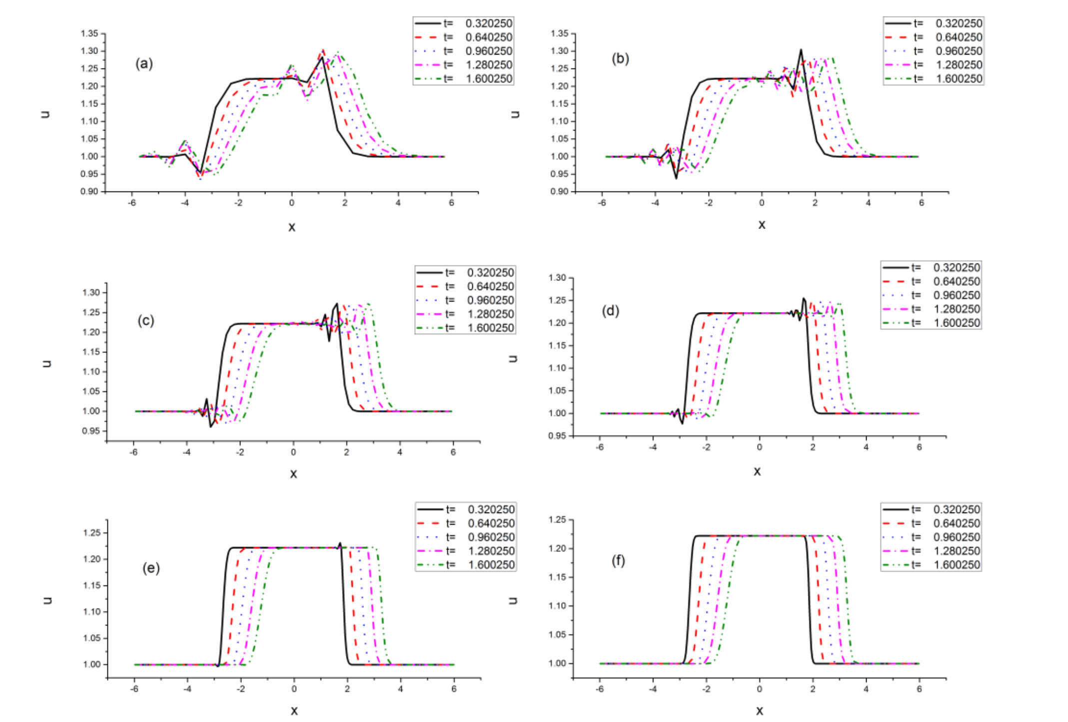

**Figure 5:** Central differencing (*U*=1, $\alpha = 0$; (a) *M*=21; (b) *M*=41; (c) *M*=81; (d) *M*=161; (e) *M*=321; (f) Exact solution)

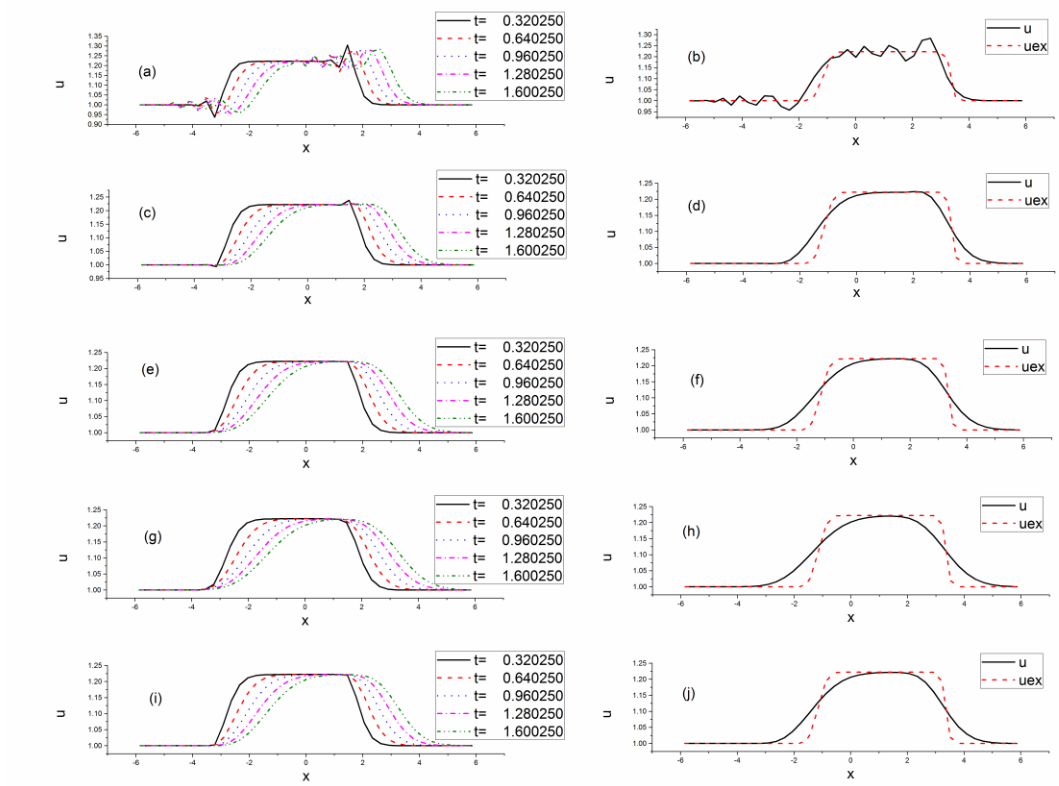

**Figure 6:** Comparison of central differencing+ $\alpha$ with upwind differencing (*U*=1, *M*=41; (a) Central, $\alpha = 0$; (b) Central, $\alpha = 0$, *t*=1.6; (c) Central. $\alpha = 0.001$; (d) Central & exact, $\alpha = 0.001$, *t*=1.6; (e) Central, $\alpha = 0.0015$; (f) Central & exact, $\alpha = 0.0015$, *t*=1.6; (g) Central, $\alpha = 0.002$; (h) Central & exact, $\alpha = 0.002$, *t*=1.6; (i) Upwind; (j) Upwind & exact, *t*=1.6)

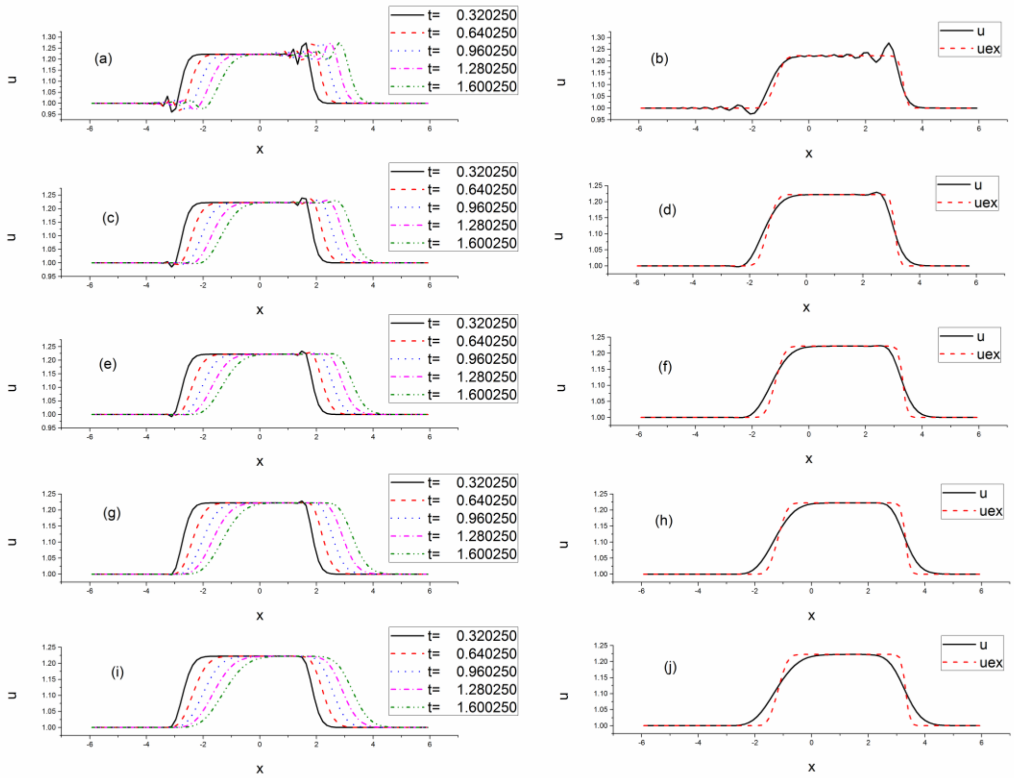


**Figure 7:** Comparison of central differencing+ $\alpha$ with upwind differencing ($U$=1, $M$=81; (a) Central differencing, $\alpha = 0$; (b) Central & exact, $\alpha = 0$, $t$=1.6; (c) Central, $\alpha = 0.001$; (d) Central & exact, $\alpha = 0.001$, $t$=1.6; (e) Central, $\alpha = 0.0015$; (f) Central & exact, $\alpha = 0.0015$, $t$=1.6; (g) Central. $\alpha = 0.002$; (h) Central & exact, $\alpha = 0.002$, $t$=1.6; (i) Upwind; (j) Upwind & exact, $t$=1.6)

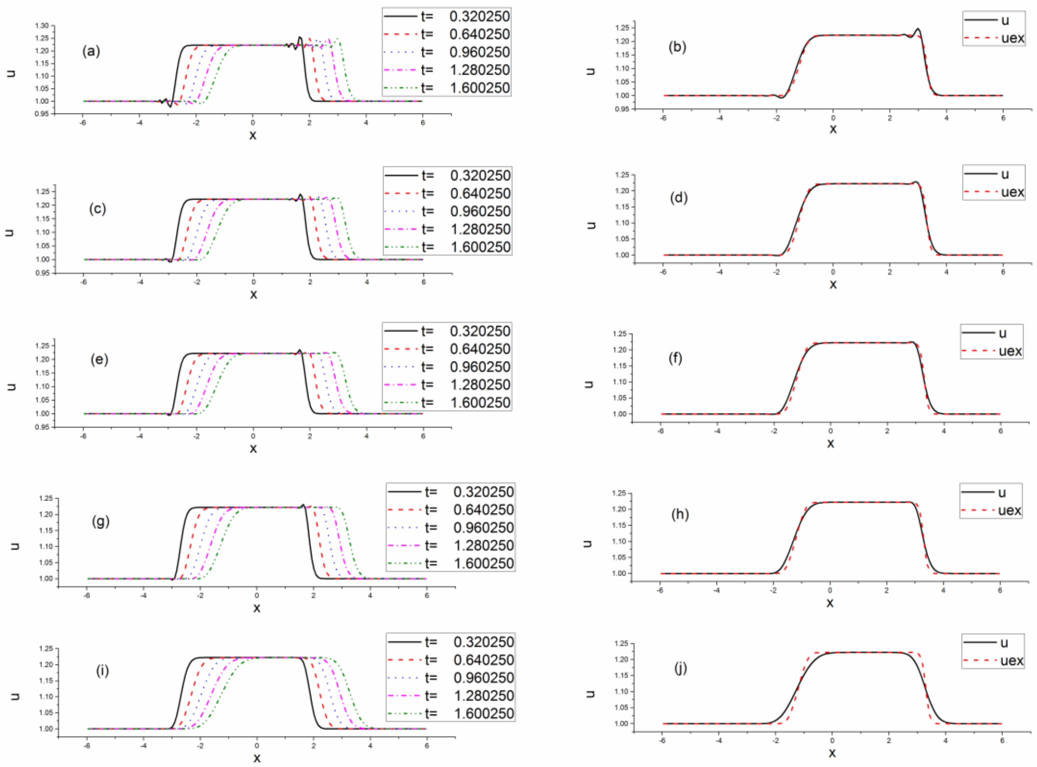


**Figure 8:** Comparison of central differencing+ $\alpha$ with upwind differencing ($U$=1, $M$=161; (a) Central, $\alpha = 0$; (b) Central & exact, $\alpha = 0$, $t$=1.6; (c) Central, $\alpha = 0.001$; (d) Central & exact, $\alpha = 0.001$, $t$=1.6; (e) Central, $\alpha = 0.0015$; (f) Central & exact, $\alpha = 0.0015$, $t$=1.6; (g) Central, $\alpha = 0.002$; (h) Central & exact, $\alpha = 0.002$, $t$=1.6; (i) Upwind; (j) Upwind & exact, $t$=1.6)

Numerical results are given in Figures 5-8. The computational condition is shown in Table 1. Figure 5 shows that if the number of division $M$ is increased, the numerical oscillations are reduced drastically. This means that if the accuracy of the difference equation is increased, the numerical oscillations would be eliminated at least in this problem. However, this results in a high cost from the viewpoint of numerical calculation. Figures 6-8 show that if $\alpha$ is increased, the numerical oscillations are reduced sufficiently. "Central differencing+ $\alpha$ " is better than Upwind differencing.

## 5. TWO-DIMENSIONAL (2D) BURGERS' EQUATION

The 2D Burger's equation is given by

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right)+f\,,\quad \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\frac{\partial}{\partial x}\left(\nu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\nu\frac{\partial v}{\partial y}\right)+g$$

$$\text{in}\quad -\infty<x<\infty\,,\ -\infty<y<\infty \tag{51a, b}$$

or

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial \nu}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial \nu}{\partial y}\frac{\partial u}{\partial y}+f\quad\text{in}\quad -\infty<x<\infty\,,\ -\infty<y<\infty\,, \tag{52a}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\frac{\partial \nu}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial \nu}{\partial y}\frac{\partial v}{\partial y}+g\quad\text{in}\quad -\infty<x<\infty\,,\ -\infty<y<\infty\,. \tag{52b}$$

In case

$$\nu=const\quad\text{and}\quad f=0\,,\quad g=0\,, \tag{53, 54}$$

we have

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),\quad \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right). \tag{55a, b}$$

We approximate the infinite region by a finite region $-L<x<L$, $-B<y<B$ and discretize the finite region as

$$dx=\frac{2L}{M}\,,\quad dy=\frac{2B}{N}\,, \tag{56a}$$

$$x_i=-L+i\,dx\,,\quad i=0,1,\cdots,M\,;\quad y_j=-B+j\,dy\,,\quad j=0,1,\cdots,N\,. \tag{56b}$$

$$u_{i\,j}^{(n)}=u(x_i,y_j,ndt)\,,\quad v_{i\,j}^{(n)}=v(x_i,y_j,ndt)\,,\quad i=0,1,\cdots,M\,,\quad j=0,1,\cdots,N\,. \tag{57}$$

The central differencing equation is given by

$$\begin{aligned}&\frac{u_{i\,j}^{(n+1)}-u_{i\,j}^{(n)}}{dt}+u_{i\,j}^{(n)}\frac{u_{i+1\,j}^{(n)}-u_{i-1\,j}^{(n)}}{2dx}+v_{i\,j}^{(n)}\frac{u_{i\,j+1}^{(n)}-u_{i\,j-1}^{(n)}}{2dy}\\&\quad=\nu\frac{u_{i+1\,j}^{(n)}-2u_{i\,j}^{(n)}+u_{i-1\,j}^{(n)}}{dx^2}+\nu\frac{u_{i\,j+1}^{(n)}-2u_{i\,j}^{(n)}+{}_{i\,j-1}^{(n)}}{dy^2}\quad\text{for}\quad i=1,2,\cdots,M-1\,,\ j=1,2,\cdots,N-1\,,\end{aligned} \tag{58a}$$

$$\begin{aligned}&\frac{v_{i\,j}^{(n+1)}-v_{i\,j}^{(n)}}{dt}+u_{i\,j}^{(n)}\frac{v_{i+1\,j}^{(n)}-v_{i-1\,j}^{(n)}}{2dx}+v_{i\,j}^{(n)}\frac{v_{i\,j+1}^{(n)}-v_{i\,j-1}^{(n)}}{2dy}\\&\quad=\nu\frac{v_{i+1\,j}^{(n)}-2v_{i\,j}^{(n)}+v_{i-1\,j}^{(n)}}{dx^2}+\nu\frac{v_{i\,j+1}^{(n)}-2v_{i\,j}^{(n)}+v_{i\,j-1}^{(n)}}{dy^2}\quad\text{for}\ i=1,2,\cdots,M-1\,,\ j=1,2,\cdots,N-1\,.\end{aligned} \tag{58b}$$

In case of the upwind differencing, the derivatives in the advection terms or the second and third terms on the left-hand side of Eq. (55) are replaced by the upwind derivatives. Namely

$$\frac{f_{i+1\,j}^{(n)}-f_{i-1\,j}^{(n)}}{2dx}\to\begin{cases}\dfrac{f_{i\,j}^{(n)}-f_{i-1\,j}^{(n)}}{dx} & \text{if } u_{i\,j}^{(n)}>0\\[2ex] \dfrac{f_{i+1\,j}^{(n)}-f_{i\,j}^{(n)}}{dx} & \text{if } u_{i\,j}^{(n)}<0\,,\\[2ex] \dfrac{f_{i+1\,j}^{(n)}-f_{i-1\,j}^{(n)}}{2dx} & \text{otherwise}\end{cases}\quad \frac{f_{i\,j+1}^{(n)}-f_{i\,j-1}^{(n)}}{2dy}\to\begin{cases}\dfrac{f_{i\,j}^{(n)}-f_{i\,j-1}^{(n)}}{dy} & \text{if } v_{i\,j}^{(n)}>0\\[2ex] \dfrac{f_{i\,j+1}^{(n)}-f_{i\,j}^{(n)}}{dy} & \text{if } v_{i\,j}^{(n)}<0\,,\\[2ex] \dfrac{f_{i\,j+1}^{(n)}-f_{i\,j-1}^{(n)}}{2dy} & \text{otherwise}\end{cases} \tag{59a, b}$$

where $f$ is $u$ or $v$. If we apply Euler solution, $u_i^{(n+1)}$ and $v_i^{(n+1)}$ are calculated from $u_i^{(n)}$ and $v_i^{(n)}$ explicitly.

The initial condition is given by

$$u_{i\,j}^{(0)}=\frac{3}{4}-\frac{1}{4}\frac{1}{1+\exp\left((-4x_i+4y_j)/(32\nu)\right)}\,,\quad v_{i\,j}^{(0)}=\frac{3}{4}+\frac{1}{4}\frac{1}{1+\exp\left((-4x_i+4y_j)/(32\nu)\right)}\,. \tag{60a, b}$$

The exact solution of this problem [10, 11] is given by

$$u(x,y,t)=\frac{3}{4}-\frac{1}{4}\frac{1}{1+\exp((-4x+4y-t)/(32\nu))}, \quad v(x,y,t)=\frac{3}{4}+\frac{1}{4}\frac{1}{1+\exp((-4x+4y-t)/(32\nu))}. \tag{61a, b}$$

In order to reduce the numerical oscillations, after $u_i^{(n+1)}$ and $v_i^{(n+1)}$ are calculated from $u_i^{(n)}$ and $v_i^{(n)}$ by Eq. (58) at the end of time step $n$, $u_i^{(n+1)}$ and $v_i^{(n+1)}$ are modified by

$$u_{i\,j}^{(n+1)}-\alpha\left[u_{i\,j}^{(n+1)}-\frac{1}{5}\left(u_{i-1\,j}^{(n+1)}+u_{i\,j}^{(n+1)}+u_{i+1\,j}^{(n+1)}+u_{i\,j-1}^{(n+1)}+u_{i\,j+1}^{(n+1)}\right)\right]\to u_{i\,j}^{(n+1)}, \tag{62a}$$

$$v_{i\,j}^{(n+1)}-\beta\left[v_{i\,j}^{(n+1)}-\frac{1}{5}\left(v_{i-1\,j}^{(n+1)}+v_{i\,j}^{(n+1)}+v_{i+1\,j}^{(n+1)}+v_{i\,j-1}^{(n+1)}+v_{i\,j+1}^{(n+1)}\right)\right]\to v_{i\,j}^{(n+1)}, \tag{62b}$$

where the quantities in the square brackets are the rapidly varying components in the signal $u_i^{(n+1)}$ and $v_i^{(n+1)}$ generated by subtracting slowly varying components or the moving averages from the signal $u_i^{(n+1)}$ and $v_i^{(n+1)}$. Rewriting, we have

$$u_{i\,j}^{(n+1)}+\frac{1}{5}\alpha\left[\left(u_{i-1\,j}^{(n+1)}-2u_{i\,j}^{(n+1)}+u_{i-1\,j}^{(n+1)}\right)+\left(u_{i\,j-1}^{(n+1)}-2u_{i\,j}^{(n+1)}+u_{i\,j+1}^{(n+1)}\right)\right]\to u_{i\,j}^{(n+1)}, \tag{63a}$$

$$v_{i\,j}^{(n+1)}+\frac{1}{5}\beta\left[\left(v_{i-1\,j}^{(n+1)}-2v_{i\,j}^{(n+1)}+v_{i-1\,j}^{(n+1)}\right)+\left(v_{i\,j-1}^{(n+1)}-2v_{i\,j}^{(n+1)}+v_{i\,j+1}^{(n+1)}\right)\right]\to v_{i\,j}^{(n+1)}. \tag{63b}$$

Equation (63) corresponds to

$$u(x,y,t)-\alpha'\left(\frac{\partial^2 u(x,t)}{\partial x^2}+\frac{\partial^2 u(x,t)}{\partial y^2}\right)\to u(x,y,t). \tag{64a}$$

$$v(x,y,t)-\beta'\left(\frac{\partial^2 v(x,t)}{\partial x^2}+\frac{\partial^2 v(x,t)}{\partial y^2}\right)\to v(x,y,t). \tag{64b}$$

### *5.1 Example 1*

Numerical results are given in Figures 9 and 10. The computational condition is shown in Table 2. Central differencing is better than the upwind differencing in this case. In this case, the viscosity coefficient $\nu$ is 0.01, and a good result is obtained even when $\alpha$ is zero.

**Table 2:** Computational condition

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $L$ | 2 | $B$ | 2 | $M$ | 81 | $N$ | 81 |
| $\nu$ | 0.01 | $dt$ | 0.00025 | $\alpha$ | 0 | | |

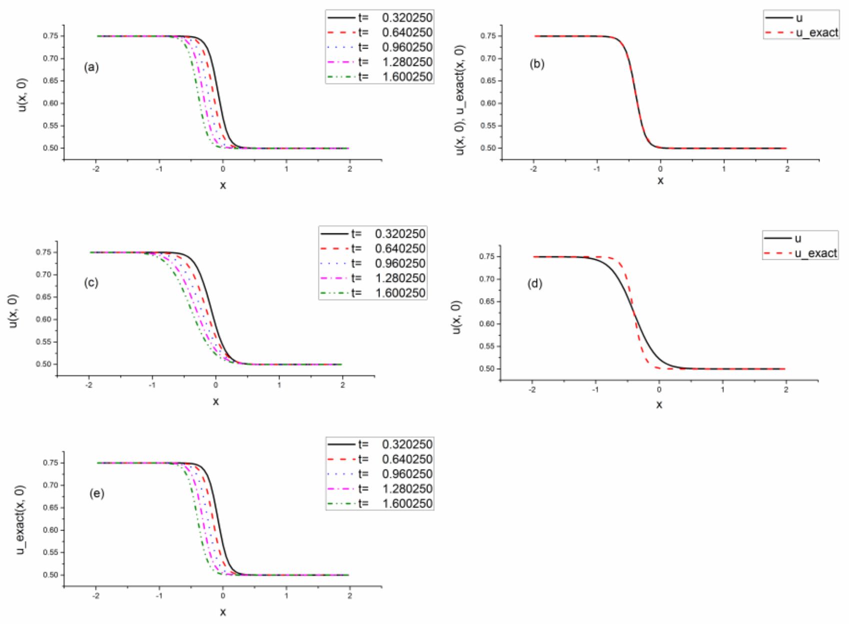


**Figure 9:** Comparison of u by central differencing with u by upwind differencing ($M$=$N$=81, $\nu=0.01$; (a) Central, $\alpha=0$; (b) Central & exact, $\alpha=0$, $t$=1.6; (c) Upwind; (d) Upwind & exact, $t$=1.6; (e) Exact solution)

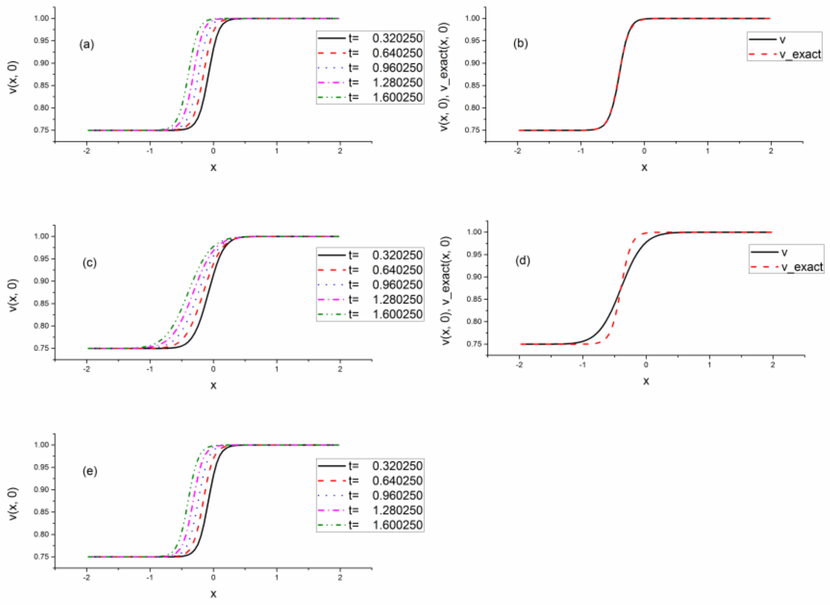


**Figure 10:** Comparison of v by central differencing with v by upwind differencing (*M*=*N*=81, $\nu = 0.01$; (a) Central, $\alpha = 0$; (b) Central, $\alpha = 0$, *t*=1.6; (c) Upwind; (d) Upwind, *t*=1.6; (e) Exact solution)

### *5.2 Example 2*

**Table 3:** Computational condition

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $L$ | 2 | $B$ | 2 | $M$ | 81 | $N$ | 81 |
| $\nu$ | 0.002 | $dt$ | 0.001 | $\alpha$ | 0, 0.004 | | |

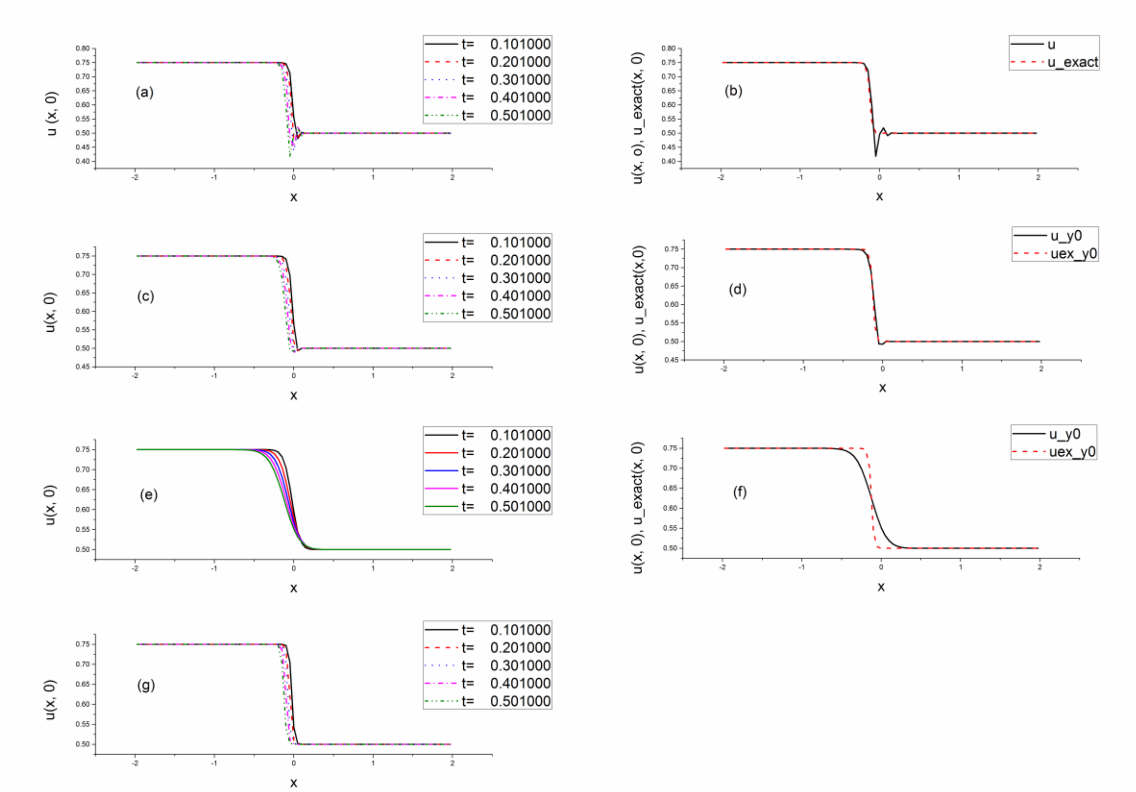


**Figure 11:** Comparison of $u$ by central differencing+$\alpha$ with $u$ by upwind differencing (*M*=*N*=81, $\nu = 0.002$; (a) Central, $\alpha = 0$; (b) Central, $\alpha = 0$, *t*=0.501; (c) Central, $\alpha = 0.004$; (d) Central, $\alpha = 0.004$, *t*=0.501; (e) Upwind; (f) Upwind, *t*=0.501; (g) Exact solution)

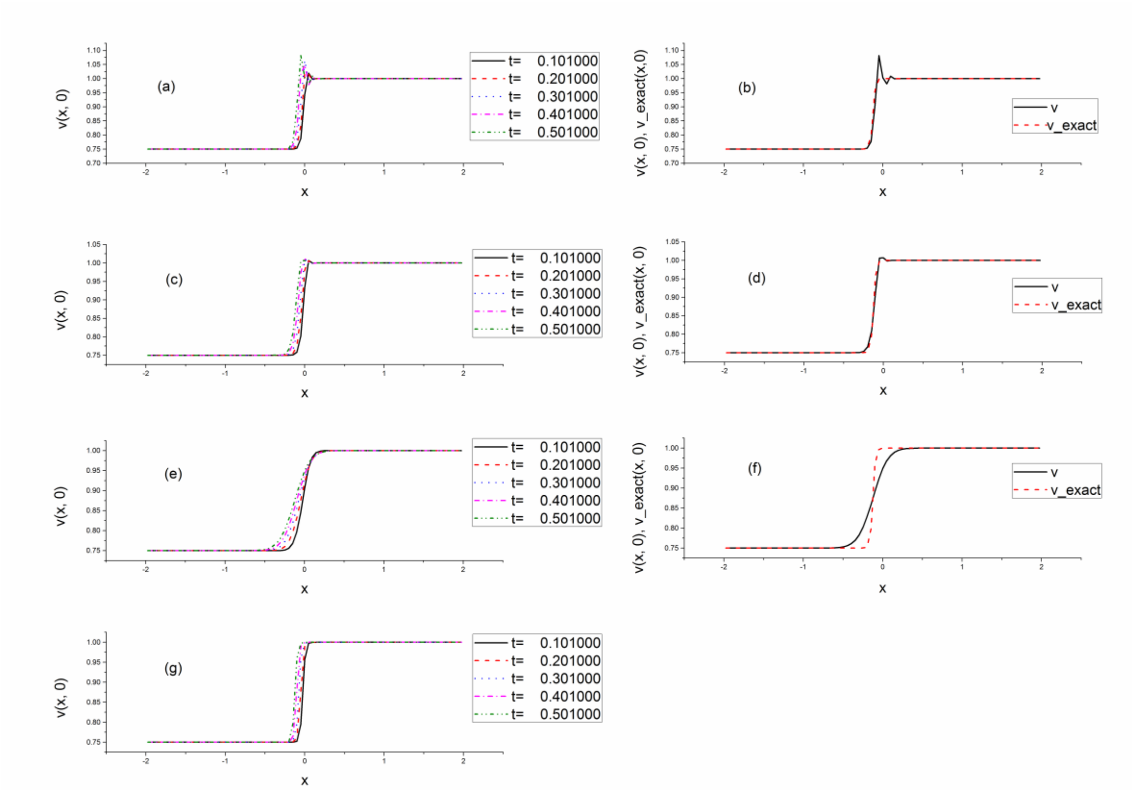

**Figure 12:** Comparison of $\nu$ by central differencing+$\alpha$ with $\nu$ by upwind differencing (*M*=*N*=81, $\nu = 0.002$; (a) Central, $\alpha = 0$; (b) Central & exact, $\alpha = 0$, *t*=0.501; (c) Central, $\alpha = 0.004$; (d) Central & exact, $\alpha = 0.004$, *t*=0.501; (e) Upwind; (f) Upwind & exact, *t*=0.501; (g) Exact solution)

Numerical results are given in Figures 11 and 12. The computational condition is shown in Table 3. In this case, the viscosity coefficient $\nu$ is 0.002, and the numerical oscillations occurs when $\alpha$ is zero. "Central differencing+$\alpha$" is much better than the upwind differencing.

## 6. CONCLUSIONS

Reduction of the spurious or numerical oscillations is very important in numerical calculations. The Upwind differencing in advection problem is well known in finite difference method (FDM), but the numerical diffusion is rather excessive. Flux limiter method overcomes the defect of the upwind differencing, but the method is not simple. In the present paper, a simple and unique method was proposed to reduce numerical oscillations effectively.

The rapidly varying component in the original signal can be made by subtracting the slowly varying component from the original signal, and the slowly varying component can be extracted by taking the moving average of the original signal. Hence, if we regard that the rapidly varying component in the original signal is the numerical oscillation, the numerical oscillation may be easily reduced or eliminated by subtracting the rapidly varying component from the original signal. This idea would easily be applied to many problems. In many cases, numerical oscillation is sinusoidal. If a signal of opposite phase is generated by differentiating the original signal twice and is added to the original signal, this kind of noise can be cancelled easily This suggests us that taking the second derivative is equivalent to generating the rapidly varying component by averaging process.

The method is verified by numerical calculations for one- and two-dimensional Burger equation. The numerical oscillation due to discontinuity of the solution is reduced effectively.

This solution can be applied to many problems and to other solutions such as FEM, BEM etc.

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